POLAR COORDINATES: WHAT THEY ARE AND HOW TO USE THEM

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1. Introduction. This note is about polar coordinates. I want to explain what they are and how to use them.

Many different coordinate systems are used in mathematics and physics and all of them share some common ideas. I think it is easier to begin by understanding what these common features are. So I am going to introduce four common ideas of coordinate systems. I will first state each idea abstractly, then illustrate it by using the usual x,y coordinates, and finally tell you how it applies to polar coordinates.

2. The coordinate system as a rule. The most basic question is: What is a coordinate system? The answer is so important that I am going to state it in bold font:

A coordinate system is a rule for mapping pairs of numbers to points in the plane.

This may not make much sense to you right now, but you'll see what I mean shortly below when we discuss the x,y and the polar coordinate systems. I do want to emphasize two things:

1. A coordinate system is not just a set of axes, it is a set of rules for mapping a pair of numbers onto a point in the plane.
2. Different coordinate systems correspond to different rules. The polar coordinate system has rules that are different than the rules of the x,y coordinate system. Other coordinate systems have yet other rules. Learning a new coordinate system comes down to understanding its rules. Keep this in mind as you read the rest of this note.

2.1. The x,y coordinates. Here are the rules for the x,y coordinate system :

1. Choose a point in the plane and call it the *origin*. The location of this point is arbitrary, i.e. you can choose any point as the origin.
2. Draw two perpendicular lines passing through the origin. These are the $x-$ and the $y-$axis. The x-axis does not have to be horizontal, nor the y-axis vertical (although that is the commonly used convention). They do have to be perpendicular. Some possible x- and y-axes are in figure 2.1a-c. Look especially at the similarity and difference between the "b" and "c" parts of the figures. The x-axis is **not** required to be horizontal – it is only a convention that the x-axis is horizontal (a convention that we will follow, but only a convention nevertheless).
3. Choose one side of the x-axis as positive. The other side of the x-axis is negative. Now rotate the positive side of the x-axis through 90 degrees counterclockwise. The part of the y-axis that it (it = the $+ve$ x-axis) coincides with is the positive y-axis. The other part of the y-axis is negative. Figure 2.2a-c shows the signs of the axes from figures 2.1a-c. Notice that once you choose

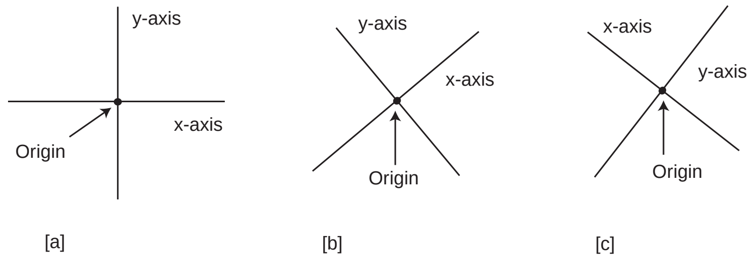


Fig. 2.1. *The coordinate axes.*

the signs on the x-axis the signs on the y-axis are completely determined by the rotation.

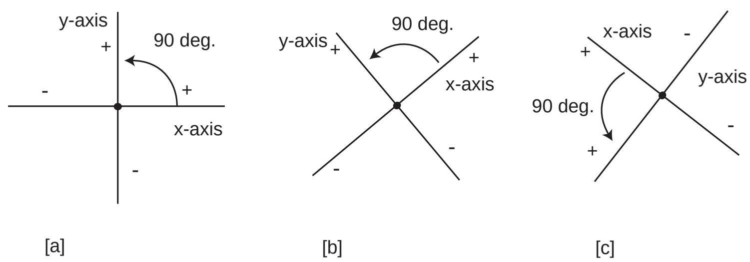


Fig. 2.2. *The coordinate axes.*

4. Having chosen an origin and the axes, here is the rule for taking a pair of numbers – say (u, v) – to a unique point in the plane (illustrated in figure 2.3, with the x- and y-axis in the conventional position). The rule is that we start from the origin, go a distance u along the x-axis and then a distance v parallel to the y-axis. Distances are considered to have signs, so that positive and negative distances on the x-axis are to the left and right of the origin, and on the y-axis towards top and bottom. The point we arrive at is the point associated with the pair of numbers (u, v). We say that (u, v) is *mapped* to this point, or that (u, v) are the *coordinates* of this point.

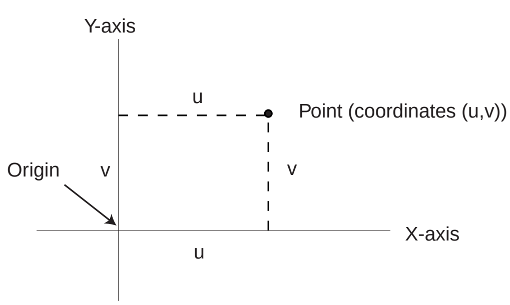


Fig. 2.3. *The xy coordinate system.*

2.2. The polar coordinates. In polar coordinates, the numbers (u, v) are interpreted **very differently** :

The first number u is taken to be a **distance** *and the second number v is taken to be an* **angle** *(usually in radians).* To be explicit about this, we will denote the pair as (r, θ) instead of (u, v). The numbers r and θ can be positive, negative or zero.

Here are the rules for the polar coordinate system:

1. Choose a point in the plane as the origin and draw the x-axis. As before, you can choose any point as the origin and the x-axis is not required to be horizontal, but is conventionally chosen to be horizontal. Mark the positive and negative sides of the x-axis with a + and a − sign as below:

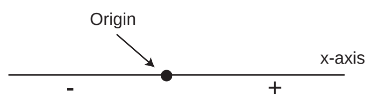


FIG. 2.4. *Origin and X-axis.*

2. Draw a line through the origin that makes an angle θ with the +ve x-axis. The angle is positive in the **counter clockwise** direction and negative in the **clockwise** direction. Call this line L:

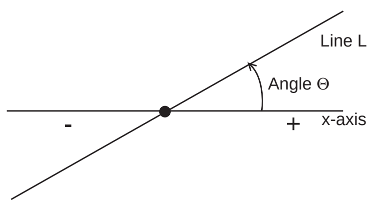


FIG. 2.5. *The Line L.*

3. Imagine rotating the x-axis through the same angle and making it coincide with the line L. Mark as positive the part of the line L that the positive x-axis coincides with and mark as negative the part that the negative x-axis coincides with. This is similar to what we did in the x,y coordinate system:

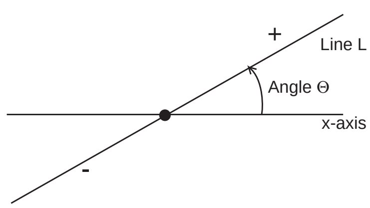


FIG. 2.6. *The signed distance along Line L.*

4. Find the point on L that is a distance r from the origin. Positive and negative distances are in those parts of L that we marked positive and negative above (figure 2.7). The point that you marked is the point that corresponds to (r, θ) in the polar coordinate system.

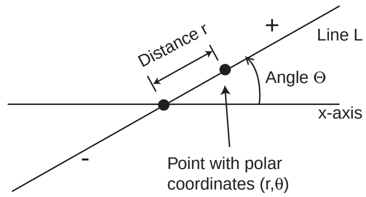
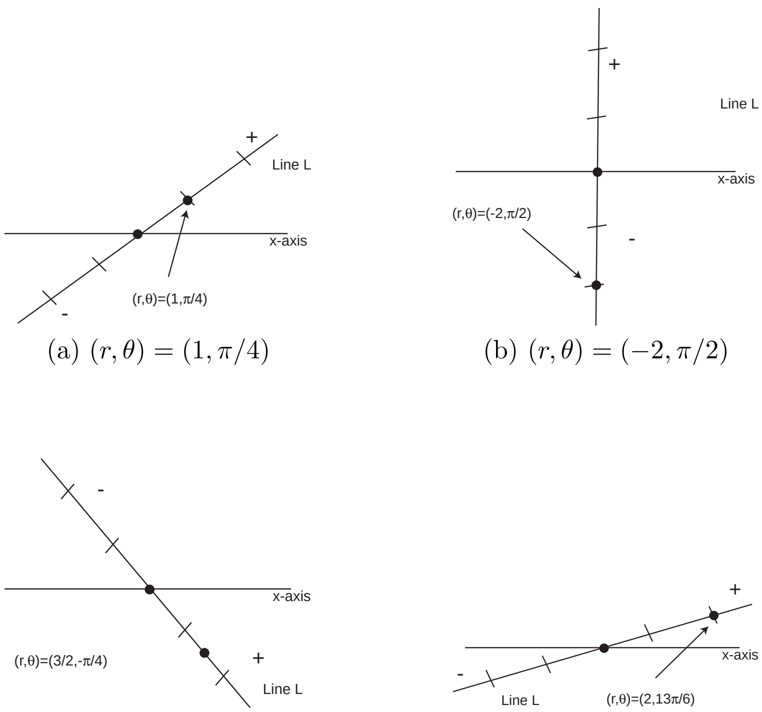


FIG. 2.7. *The point with polar coordinates* (r, θ).

That's it. That's the rule for polar coordinates. The numbers (r, θ) are called the *polar coordinates* of the point we plotted.

2.3. Examples. Below are some examples of plotting points using their polar coordinates. Please try to do the examples yourself and compare the results. Keep in mind that all angles are in radians. Be sure that you can do and understand the examples c-d (Hint: $13\pi/6 = 2\pi + \pi/6$).



(a) $(r, \theta) = (1, \pi/4)$

(b) $(r, \theta) = (-2, \pi/2)$

(c) $(r, \theta) = (3/2, -\pi/4)$

(d) $(r, \theta) = (2, 13\pi/6)$

FIG. 2.8. *Examples*

2.4. Some properties of polar coordinates. There are some aspects of polar coordinates that are tricky. You should pay attention to the following:

1. Two different polar coordinates, say (r_1, θ_1) and (r_2, θ_2), can map to the <u>same</u> point. This can happen in the following ways:
   (a) It can happen if $r_2 = r_1$ and $\theta_2 = \theta_1 \pm 2\pi n$ for any non zero integer n. The angle $2\pi n$ corresponds to n complete rotations, counter clockwise for n positive and clockwise for n negative. Hence, the lines L corresponding to θ_1 and $\theta_2 = \theta_1 \pm 2\pi n$ are the same and have the same positive and negative parts. Going the same distance $r_1 = r_2$ gets to the same point.

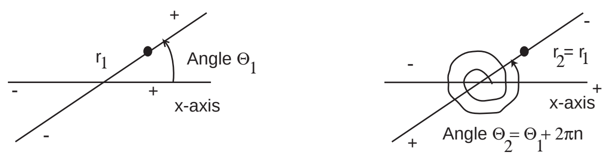


FIG. 2.9. *The condition* $r_2 = r_1$ *and* $\theta_2 = \theta_1 \pm 2\pi n$.

   (b) (This may be little difficult to understand the first time.) It can happen if $r_2 = -r_1$ and $\theta_2 = \theta_1 \pm \pi \pm 2\pi n$ for any non zero integer n (positive or negative). The lines corresponding to θ_1 and $\theta_2 = \theta_1 \pm \pi$ (and hence $\theta_2 = \theta_1 \pm \pi \pm 2\pi n$) have the same inclination but the positive and negative parts are switched. Therefore going the distance $r_2 = -r_1$ gives the same point. The following figure shows this. Be sure that you understand the signs on the line L.

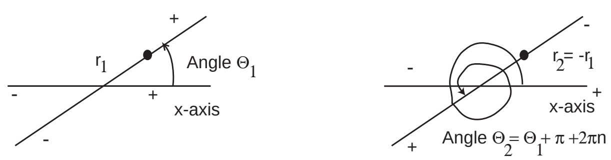


FIG. 2.10. *The condition* $r_2 = -r_1$ *and* $\theta_2 = \theta_1 \pm \pi \pm 2\pi n$.

2. All polar coordinates $(0, \theta)$ map to the same point. This is so because for any θ the point that is distance 0 away from the origin along the line L is the origin:

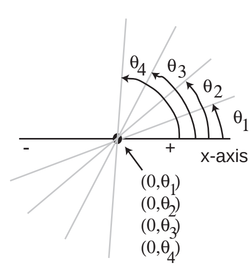


FIG. 2.11. *The condition* $(0, \theta)$.

2.5. Restricted polar coordinates. Is it possible to modify our definition of polar coordinates so that different coordinates do not map to the same point? It is certainly possible to modify the definition so as to get around 1 (a) and (b) above. But getting around 2 is impossible as long as we want to have polar coordinates for the origin.

Let us see what we can do with 1 (a) and (b). The reason multiple values of θ lead to the same point is that as the line L rotates, it coincides with itself at multiples of π radians. When these are even multiples, the positive part of L coincides with the positive part, when these are odd multiples the positive part coincides with the negative part.

We can "fix" these problems in two ways:

1. We allow θ to take only those values where the line L will not coincide with itself. That is, we allow θ to only lie in any one of the following range

$$\begin{aligned} 0 &\leq \theta < \pi, \text{ or,} \\ 0 &< \theta \leq \pi, \text{ or,} \\ -\pi &\leq \theta < 0, \text{ or,} \\ -\pi &< \theta \leq 0. \end{aligned}$$

   The value of r, of course, can be negative, zero, or positive.
   Please draw a rough sketch and convince yourself that the above restrictions on θ do in fact give unique polar coordinates to every point in the plane. Also, see if you can answer this question: What goes wrong if we allow the range of θ to contain both 0 and (plus or minus) π?
2. Restrict θ to one of the ranges

$$\begin{aligned} 0 &\leq \theta < 2\pi, \text{ or,} \\ 0 &< \theta \leq 2\pi, \text{ or,} \\ -2\pi &\leq \theta < 0, \text{ or,} \\ -2\pi &< \theta \leq 0. \end{aligned}$$

   and restrict r to nonnegative values (i.e. $r \geq 0$).
   Again draw a sketch and see that this works.

As I said above, fixing the problem 2 described in the previous section is impossible if we want polar coordinates for the origin.

With this discussion of restricted polar coordinates, we have finished the first point of this note – to understand what the polar coordinate system is.

3. The coordinate grid. We now come to the next basic idea – that of a *coordinate grid*. The coordinate grid is a tool for simultaneously visualizing coordinates of all points in the plane. This mental picture of the coordinate grid is a very useful.

In the x,y coordinate system, the grid consists of a number of lines. Along each line, only one coordinate varies while the other remains constant. For example, along any grid line that runs parallel to the y axis, the x coordinate is fixed. And along any grid line that runs parallel to the x axis, the y coordinate is fixed.

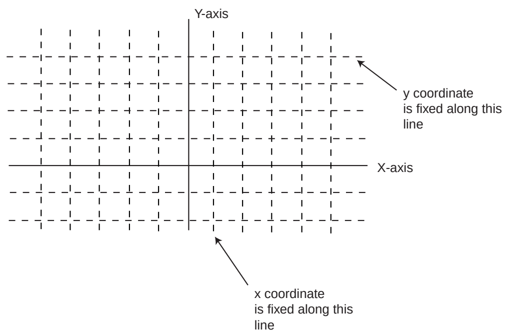


FIG. 3.1. *The xy coordinate grid.*

3.1. The polar coordinate grid. Let's apply this idea to polar coordinates.

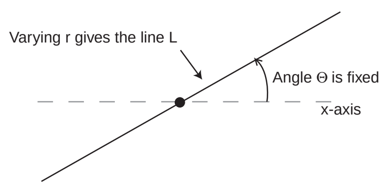


FIG. 3.2. *Fixed θ, variable r.*

First, let's hold θ fixed, and vary r. This just gives us the line L. Then, we change θ, fix it again, and vary r. This gives another line L. Thus, holding θ fixed at different values and varying r gives us a pattern of rays spreading out from the origin:

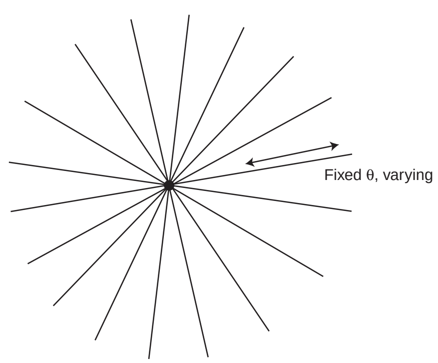


FIG. 3.3. *Rays through the origin formed by fixing θ and varying r.*

Next, let's hold r fixed and vary θ. This gives a circle of radius r with the origin as the center:

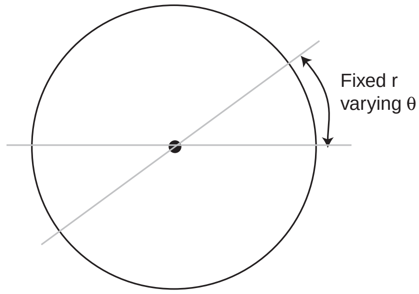


FIG. 3.4. *Fixed r, varying θ.*

Changing the values of r and repeating this procedure gives a set of concentric circles:

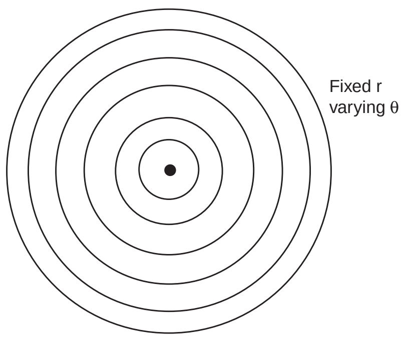


FIG. 3.5. *Fixed r at different values, varying θ.*

Putting the rays and the concentric circles together we get the polar coordinate grid:

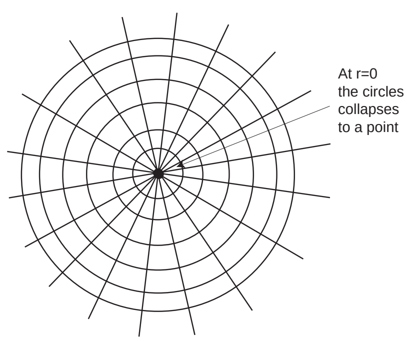


FIG. 3.6. *The polar coordinate grid.*

Note that the circles collapse to a point at the origin. This is really just the visual manifestation of the fact that the point $(0, \theta)$ is the origin for any θ, .

The collapse of the circle to a point at the origin turns out to be a serious problem in many applications of polar coordinates and you should forever be alert to this issue.

This ends the discussion of the coordinate grid.

4. Relation between polar and x,y coordinate systems. We now have two coordinate systems (the x,y and the polar) and a natural question to ask is: what is the relation between them? This is the third issue that we will grapple with. Suppose we choose a point in the plane as the origin and set up an x,y and a polar coordinate system at that origin. Then, we pick a point in the plane, say A, and calculate the x,y and the polar coordinates of A. What is the relation between the two coordinates?

The following figure shows the situation (don't forget that the angle θ is measured from the +ve x-axis):

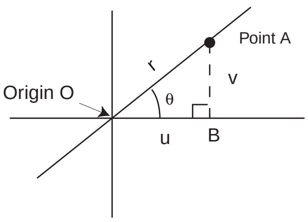


FIG. 4.1. *Conversion between coordinate systems.*

Because OAB is a right angled triangle (why?), it is easy to express the x and y coordinates (which we denote u and v respectively) in terms of the r and θ coordinates:

$$u = r\cos\theta$$
$$v = r\sin\theta.$$

From these equations we get

$$u^2 + v^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2,$$
$$\text{so that } u^2 + v^2 = r^2, \text{ and } r = \sqrt{u^2 + v^2}.$$

Similarly,

$$\frac{v}{u} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta, \text{ so that } \theta = \tan^{-1}\frac{v}{u}.$$

Thus, we can write r and θ in terms of x and y as:

$$r = \sqrt{u^2 + v^2},$$
$$\theta = \tan^{-1}\frac{v}{u}.$$

That's all there is to the relation between the two coordinate systems.

4.1. Graphing a function. The final important idea is that of graphing a function. Again, let us start with the familiar x,y coordinate system.

To graph the function $y = f(x)$, we first find the *domain* of the function. The domain of the function is the range of values of x for which the function can be used. Sometimes the domain is given explicitly, e.g.

$$0 \le x \le 3.$$

On other occasions, you have to figure out the domain from the function itself.

For example, take the function

$$f(x) = \sqrt{x}.$$

This function is meaningless if $x < 0$ (negative numbers do not have square roots). Therefore, its domain is all $x \geq 0$.

After figuring out the domain of the function, you figure out whether or not the function approaches $\pm\infty$ anywhere. Typically the function will approach $\pm\infty$ at those points where division by zero occurs (these are points that you have excluded from the origin in the previous step), e.g. the function $f(x) = \frac{1}{(x-2)}$ is not defined for $x = 2$ and as x approaches 2 from the left $f(x)$ goes to $-\infty$ and as x approaches 2 from the right $f(x)$ goes to $+\infty$.

Having determined the domain and the points where the function goes to $\pm\infty$, you plot the graph between these points. You pick any number x in the domain, calculate $y = f(x)$, and plot the pair (x, y). This gives you one point in the coordinate plane. Plotting such points for all values x in the domain of the function gives the graph of the function.

In practice, this is impossible to do by hand if the domain of the function contains infinite values of x. But in many simple cases, we can plot a few points and connect them smoothly with a curve.

As an example, below I draw the graph of the function $f(x) = \sqrt{x}$ by hand. To do this, I first notice that the domain of f is $x \geq 0$ and there are no points x where the function approaches $\pm\infty$. I then pick $0, 1, 4, 9$ as values of x, calculate the corresponding values of y and enter them in a table like this:

	x	$y = f(x) = \sqrt{x}$
1	0	0
2	1	1
3	4	2
4	9	3

I make the table like this: The first column simply contains labels that I give to the points I am going to plot. The second column has the values of x, and the final column has values of $f(x)$. Then, I plot the (x, y) pairs in all the rows of the table, indicate the labels, and connect the points with a smooth curve that connects the labels in the sequence $1, 2, 3, \ldots$ (figure 4.2). This is a sketch of the graph of $f(x) = \sqrt{x}$.

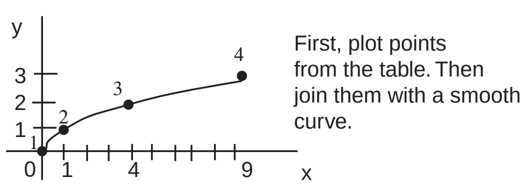


FIG. 4.2. *Graph of* $y = f(x) = \sqrt{x}$.

4.2. Graphing polar functions. We now follow the same idea for graphing polar functions. Polar functions are usually specified as $r = f(\theta)$. Let's start with a polar version of the previous example. Let's plot

$$r = \sqrt{\theta}.$$

First, note that the domain of the function is $\theta \geq 0$. Next, we create a table for values of r, θ. To make our plotting easy, we'll start by taking some simple angles in degrees, convert them to radians and enter them in the table.

As before, the first column of the table simply contains labels for the points. The labels are $1, 2, \ldots$. Then I choose values of θ from 0 degrees and increment them by 45 degrees. I enter these in the second column as radians and degrees. The third column shows the value of $r = f(\theta)$:

	θ	$r = f(\theta) = \sqrt{\theta}$
1	0 (0 degrees)	0
2	$\pi/4 \simeq 0.785$ (45 degrees)	0.89
3	$\pi/2 \simeq 1.571$ (90 degrees)	1.25
4	$3\pi/4 \simeq 2.356$ ($135 = 90 + 45$ degrees)	1.53
5	π (180 degrees)	1.77
6	$5\pi/4 \simeq 3.927$ ($225 = 180 + 45$ degrees)	1.98
7	$3\pi/2 \simeq 4.712$ (270 degrees)	2.17
8	$7\pi/4 \simeq 5.498$ ($315 = 270 + 45$ degrees)	2.34
9	$2\pi \simeq 6.283$ (360 degrees)	2.51

I plot these points one by one using the polar coordinates r, θ from the table. I also indicate the label of each point. Here is the plot:

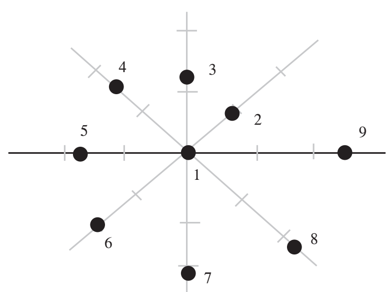


FIG. 4.3. *Points for plotting the graph of* $r = f(\theta) = \sqrt{\theta}$.

Next, I connect starting from the first point (labeled 1) in sequence $(2, 3, 4, \ldots)$ to the last by a smooth curve :

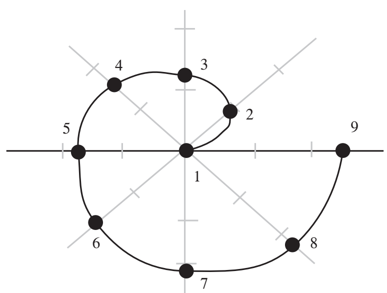


FIG. 4.4. *Connected points for plotting the graph of* $r = f(\theta) = \sqrt{\theta}$.

Common Pitfall: Students sometimes connect the first points by a radial straight line like this:

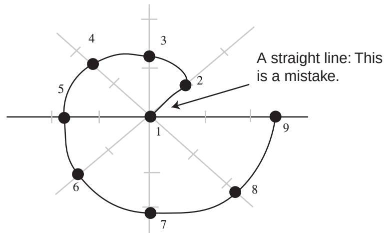


FIG. 4.5. *A common mistake in drawing polar graphs.*

This is a mistake!! To understand how the polar graph looks between the first two points, note that for $\theta = \pi/8$ radians $= 45/2$ degrees the value of r is $r = \sqrt{\pi/8} \simeq= 0.63$ so that if we were to plot this point the curve would look like:

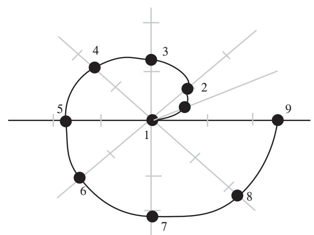


FIG. 4.6. *Including the point at $\pi/8$.*

In fact, you can never get a radial straight line in a polar plot of any function $r = f(\theta)$, because a radial straight line implies that there are multiple values of r at that θ and that can never happen for a function $r = f(\theta)$ (a function always gives a single output number for a single input number).

If you find ourself drawing a radial straight line between two points that you have plotted, pause and think about what the value of r would be *between* the two angles.

4.3. Plotting (contd.). Returning back to figure 4.4, recall that we have only plotted values for θ between 0 and 2π. Because $r = \sqrt{\theta}$ is an increasing function of θ, as θ increases beyond 2π, the value of r will keep increasing and the plot will look like a spiral that winds around the origin while getting farther and farther away from the origin:

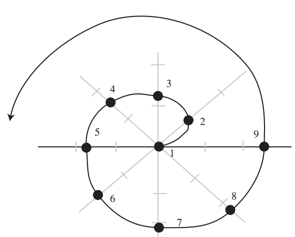


FIG. 4.7. *The graph of $r = f(\theta) = \sqrt{\theta}$.*

4.4. More complicated functions. The plotting procedure can get more complicated if the function can go to $\pm\infty$. Let's understand this in x, y coordinates first. Take the function

$$f(x) = \frac{1}{(x-2)}$$

again. As we said before, the domain of this function is all numbers except $x = 2$ (because division by 0 is undefined). So let us plot the function separately for $x > 2$ and $x < 2$. Here is the table of values for $x > 2$:

x	$y = f(x) = \frac{1}{(x-2)}$
2.5	2
3	1
4	0.5
5	0.33

A little thought should convince you that that x approaches 2 from the right $f(x)$ goes to $+\infty$ and as x goes to $+\infty$, $f(x)$ goes to zero. We'll add these two values to our table. And we add labels as well:

	x	$y = f(x) = \frac{1}{(x-2)}$
1	$x \to 2+$	$+\infty$
2	2.5	2
3	3	1
4	4	0.5
5	5	0.33
6	$x \to +\infty$	0

Next we make a similar table for $x < 2$

	x	$y = f(x) = \frac{1}{(x-2)}$
7	$x \to 2-$	$-\infty$
8	1.5	-2
9	1	-1
10	0	-0.5
11	-1	-0.33
12	$x \to -\infty$	0

And then we sketch the function as follows:

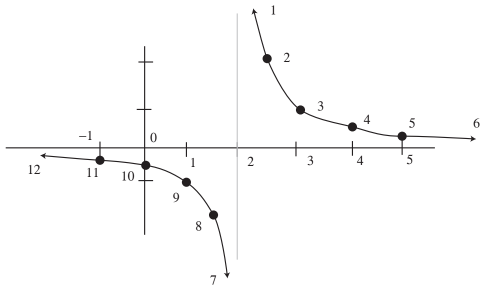


FIG. 4.8. *The graph of* $y = f(x) = \frac{1}{(x-2)}$.

4.5. More complicated polar function. Next we plot polar function that goes to ∞:

$$r = f(\theta) = \frac{1}{\tan\theta}, \quad \text{for } -\pi/2 < \theta < +\pi/2.$$

First note that we are given the range in which to plot the function. Next note that $\tan 0 = 0$ and since division by 0 is meaningless we will have to exclude the point $\theta = 0$ from the domain of the function. Thus, we need two tables, the first for $-\pi/2 < \theta < 0$ and the second for $0 < \theta < +\pi/2$. In each table we follow the strategy of incrementing the angle by 45 degrees. Here is the first table:

	θ	$r = f(\theta) = \frac{1}{\tan\theta}$
1	$\theta \to -\pi/2+$	0
2	$-\pi/4$	-1
3	$\theta \to 0-$	$-\infty$

Here is the second table

	θ	$r = f(\theta) = \frac{1}{\tan\theta}$
4	$\theta \to 0+$	$+\infty$
5	$\pi/4$	1
6	$\theta \to \pi/2-$	0

Next we plot the values in both tables:

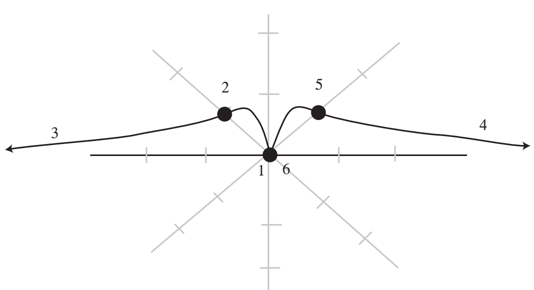


FIG. 4.9. *The graph of* $r = f(\theta) = \frac{1}{\tan\theta}$.

4.6. Recap. That's all there is to plotting in polar coordinates:

1. From the equation $r = f(\theta)$ find the domain of the function. Or else, check whether the domain is given in the problem. The domain is the set of values of θ for which the calculation of $f(\theta)$ makes sense.
2. Check if the function goes to $\pm\infty$. If so, note down where it does.
3. Pick some angles for which it is easy to draw the lines L. You can pick them in degrees to start, but you must convert them into radians before you plug them into the formula for f. I usually start from 0 degrees and increment by 45 degrees till I get to 360 degrees. Sometimes the increment of 45 degrees is too much and I cannot figure out what the graph looks like. Then I go back and calculate some values between the 45 degree angles.
4. Make a table of θ and $r = f(\theta)$ for the above values of θ.
5. Plot the points r, θ in the table using polar coordinates.
6. Join the points with a smooth curve (avoid radial straight lines).

You should practice plotting by using this procedure explicitly. Later on you can take short cuts, but initially don't omit any steps.

5. Graphing periodic functions. Problem sets in graphing polar functions often have trigonometric functions that are periodic (e.g. sin and cos). You should pay special attention to graphing such functions, and I am writing this section to help you with them.

We'll start with a simple problem: graph $r = f(\theta) = 1 + \cos\theta$.

First, we determine the domain. We know that $\cos\theta$ is defined for all values of θ, so the domain of the function is all real numbers. But there is an interesting wrinkle to this problem. The function $\cos\theta$ is periodic, with a period of 2π. So $1 + \cos\theta$ is also periodic with period 2π:

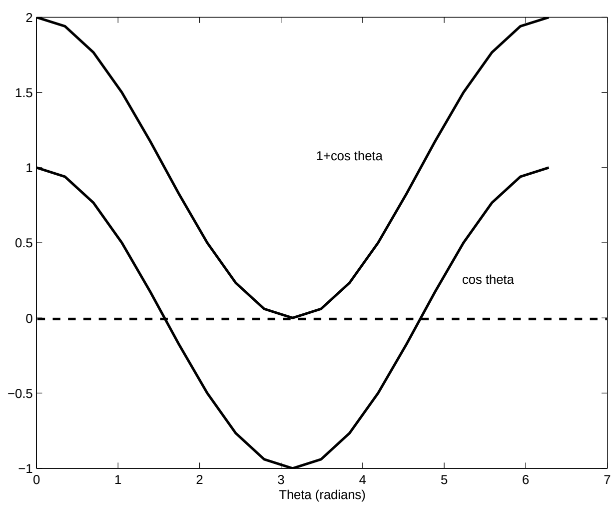


FIG. 5.1. *The functions* $\cos\theta$ *and* $1 + \cos\theta$.

This suggests that we only need plot the graph for θ between 0 and 2π. Beyond these limits the graph will simply repeat itself.

Next, we make a table for the values we are going to plot. The first column shows labels of points as usual. The second column shows values of θ in degrees (I chose these values because it is easy to plot lines L at these angles and also because we know the values of cos for these angles). The third column shows that values of θ in radians and the fourth and fifth columns show the values of $\cos\theta$ and $1 + \cos\theta$:

	θ (degrees)	θ (radians)	$\cos\theta$	$r = f(\theta) = 1 + \cos\theta$
1	0	0	1	2
2	45	$\pi/4$	$\frac{1}{\sqrt{2}} \simeq 0.71$	1.71
3	90	$\pi/2$	0	1
4	135 $(= 90 + 45)$	$3\pi/4$	$-\frac{1}{\sqrt{2}} \simeq -0.71$	0.29
5	180	π	-1	0
6	225 $(= 180 + 45)$	$5\pi/4$	$-\frac{1}{\sqrt{2}} \simeq -0.71$	0.29
7	270	$3\pi/2$	0	1
8	$315 = 270 + 45$	$7\pi/8$	$\frac{1}{\sqrt{2}} \simeq 0.71$	1.71
9	360	2π	1	2

Notice that the last line in the table is the same as the first line – this is a consequence of cosine being periodic with a period of 2π.

Finally, we plot the points and join them to get the graph:

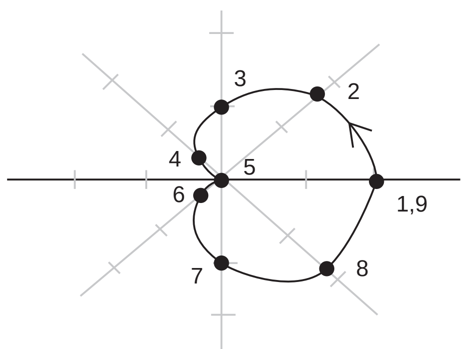


FIG. 5.2. *Graph of* $r = f(\theta) = 1 + \cos\theta$.

This "heart-shaped" curve is called the *cardioid* (pronounced car-d-oyd).

5.1. More periodic functions. The example we just did was rather simple – the argument of cos was just θ so we knew that the period was 2π, and the values of r were all non-negative. Plotting gets more complicated when these conditions do not hold.

5.1.1. Functions of $m\theta$. Let's first consider what happens when the argument is a multiple of θ. Take the function $r = f(\theta) = 1 + \cos 2\theta$. The argument of cos is 2θ. Since we know the values of cos for $0, 45, 90, \ldots$ degrees we should set 2θ to these numbers. Accordingly, I first write the values of 2θ in degrees in the table, then calculate θ and $1 + \cos 2\theta$.

	2θ (degrees)	θ (degrees)	$\cos 2\theta$	$r = f(\theta) = 1 + \cos 2\theta$
1	0	0	1	2
2	45	22.5	$\frac{1}{2} \simeq 0.71$	1.71
3	90	45	0	1
4	135	67.5	$-\frac{1}{2} \simeq -0.71$	0.29
5	180	90	-1	0
6	225	112.5	$-\frac{1}{2} \simeq -0.71$	0.29
7	270	135	0	1
8	315	157.5	$\frac{1}{2} \simeq 0.71$	1.71
9	360	180	1	2

Remember that we have to plot r and θ. In this table, θ only goes from 0 to 180. We need values of θ till 360 degrees to get a complete plot. So let's keep incrementing 2θ by 45 degrees till we get θ equal to 360 degrees (I am repeating the last row of the previous table as the first row of the next table):

	2θ (degrees)	θ (degrees)	$\cos 2\theta$	$r = f(\theta) = 1 + \cos 2\theta$
9	360	180	1	2
10	405	202.5	$\frac{1}{2} \simeq 0.71$	1.71
11	450	225	0	1
12	495	247.5	$-\frac{1}{2} \simeq -0.71$	0.29
13	540	270	-1	0
14	585	292.5	$-\frac{1}{2} \simeq -0.71$	0.29
15	630	315	0	1
16	675	337.5	$\frac{1}{2} \simeq 0.71$	1.71
17	720	360	1	2

Now notice that the r column of this table is identical to the previous table. As we shall see below, this has an important implication.

But first, let us plot the points. Below, there are three plots. The (a) part shows the graph for θ from 0 to 180 (the first of the two tables we created), the (b) part shows the graph for θ from 180 to 360, and (c) part shows the complete graph.

Notice that the (b) part of the figure is just the (a) part rotated 180 counter clockwise:

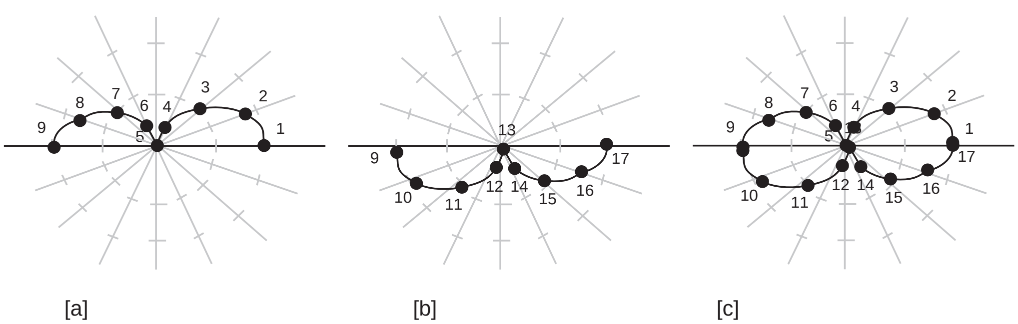


FIG. 5.3. *Graph of $r = f(\theta) = 1 + \cos 2\theta$.*

This is no coincidence. But it is a consequence of the fact that the function we have plotting has 2θ as the argument of cos. Here is how to understand it: As θ goes from 0 to 180, the argument 2θ goes from 0 to 360. As θ goes further from 180 to 360, the argument 2θ goes from 360 to 720. *But because cos has a period of* 360 *degrees, the values of* $\cos 2\theta$ *between* $0-360$ *and* $360-720$ *are identical.* That is why the second table we created has r values identical to the first table. But the θ values of the second table are shifted from the θ values of the first table by 180 degrees (please check the rows of both table to understand this). Thus the graph of the second table is the graph of the first table with 180 degrees added to θ – that is, it is rotated counter-clockwise by 180 degrees.

You should now clearly see that we can use this to plot any function of $\cos m\theta$ or $\sin m\theta$, where m is an integer (e.g. $\cos 4\theta$, $\sin 8\theta$). Proceed as follows

1. Create a table – like the one we created above – in which the second column is $m\theta$, the third column is θ and so on with the last column $r = f(\theta)$.
2. In the $m\theta$ column starting from 0 degrees increment by 45 degrees till you reach 360. These are the values of $m\theta$ that we will use.
3. From the values of $m\theta$ calculate θ (divide $m\theta$ by m).
4. Fill in all of the table using the formula for $f(\theta)$. The table should look like this:

	$m\theta$ (degrees)	θ (degrees)	$\cdots$	r
1	0	$0/m$	$\cdots$	$\cdots$
2	45	$45/m$	$\cdots$	$\cdots$
3	90	$90/m$	$\cdots$	$\cdots$
4	135	$135/m$	$\cdots$	$\cdots$
5	180	$180/m$	$\cdots$	$\cdots$
6	225	$225/m$	$\cdots$	$\cdots$
7	270	$270/m$	$\cdots$	$\cdots$
8	315	$315/m$	$\cdots$	$\cdots$
9	360	$360/m$	$\cdots$	$\cdots$

5. Now plot the points 1-9 using the θ and r values in the table. The plot will only go from 0 degrees to $360/m$ degrees.
   As an example, below is a plot for a hypothetical function with $m = 6$ might look like. I am only showing some of the radial lines L for clarity. Keep in mind that $360/6 = 60$ degrees:

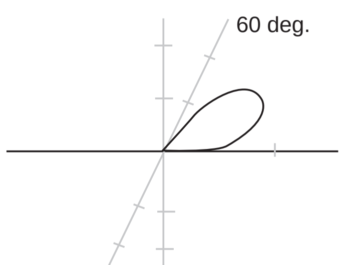


FIG. 5.4. *Polar plot of a periodic function with argument* $6m$.

6. Now rotate the plot clockwise around the origin m times in $360/m$ degree increments. This is the graph of the function:

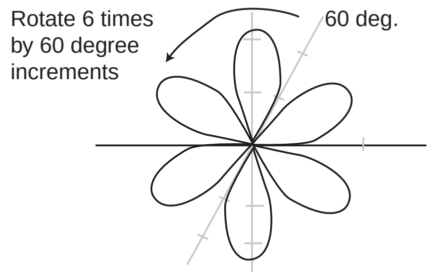


FIG. 5.5. *Complete plot of a periodic function with argument* $6m$.

5.1.2. Functions that take negative values. Next consider the function $r = f(\theta) = \frac{1}{2} + \cos\theta$. Here is the table for graphing this function:

	θ (degrees)	θ (radians)	$\cos\theta$	$r = f(\theta) = \frac{1}{2} + \cos\theta$
1	0	0	1	1.5
2	45	$\pi/4$	$\frac{1}{\sqrt{2}} \simeq 0.71$	1.21
3	90	$\pi/2$	0	0.5
4	$135\ (= 90 + 45)$	$3\pi/4$	$-\frac{1}{\sqrt{2}} \simeq -0.71$	-0.21
5	180	π	-1	-0.5
6	$225\ (= 180 + 45)$	$5\pi/4$	$-\frac{1}{\sqrt{2}} \simeq -0.71$	-0.21
7	270	$3\pi/2$	0	0.5
8	$315 = 270 + 45$	$7\pi/8$	$\frac{1}{\sqrt{2}} \simeq 0.71$	1.21
9	360	2π	1	1.5

Notice that we have negative numbers for r. The figure 5.6 on page 21 shows individual points from (a)-(i). I have marked the +ve part of the line L in each figure to help you understand the plotting of negative values of r. The complete plot is shown in figure 5.7 on page 21.

5.2. More complications. Nothing stops us from putting together everything we have learned and plotting a periodic function with cos or sin $m\theta$ which also goes negative. Proceed just as before. Take $m\theta$ from 0 to 360 degrees in 45 degree increments, for these values calculate θ and $r = f(\theta)$. Then, plot these points taking care about negative values. Finally rotate the plot counter clockwise m times till it joins itself.

Common Pitfall: The procedures from section 5.1 onwards apply **only** when the function being plotted contains periodic function of θ that depends on θ by $m\theta$. For any other function you have to fall back to the more general procedure of section 4.2.

5.3. Powers of r. There is a rare but sneaky problem that you should be aware of. In this problem, $f(\theta)$ is given as an even power of r, e.g.

$$r^2 = (1 + \cos\theta).$$

It is tempting to simplify the equation to $r = \sqrt{1 + \cos\theta}$. But this is wrong, because it is only the positive square root. The correct simplification is to consider the positive

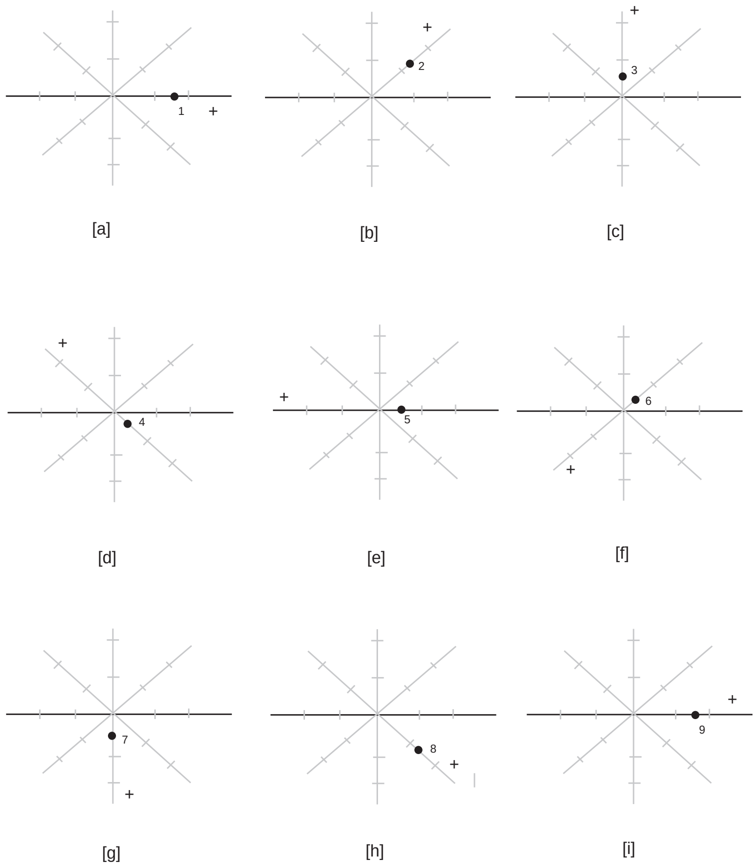


FIG. 5.6. *Plots of individual points for $r = f(\theta) = \frac{1}{2} + \cos\theta$.*

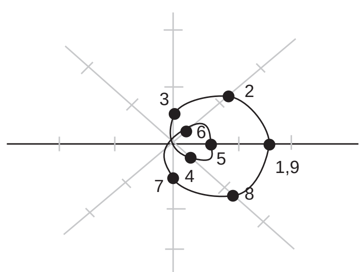


FIG. 5.7. *Plot of $r = f(\theta) = \frac{1}{2} + \cos\theta$.*

and negative square roots:

$$r = \pm\sqrt{1 + \cos\theta}.$$

This means that for every theta there are **two** values of r (in the example the values are $+\sqrt{1+\cos\theta}$ and $-\sqrt{1+\cos\theta}$). You have to plot both values of r and connect them to get the plot.

A simple modification of our procedure is sufficient to deal with this:

1. In the column for r, we enter both values of r (the positive and the negative). The table for $r = \pm\sqrt{1 + \cos\theta}$ looks like this:

	θ (degrees)	θ (radians)	$\cos\theta$	$r = f(\theta) = \sqrt{1 + \cos\theta}$
1	0	0	1	± 1.41
2	45	$\pi/4$	$\frac{1}{\sqrt{2}} \simeq 0.71$	± 1.31
3	90	$\pi/2$	0	± 1
4	$135\ (= 90 + 45)$	$3\pi/4$	$-\frac{1}{\sqrt{2}} \simeq -0.71$	± 0.54
5	180	π	-1	± 0
6	$225\ (= 180 + 45)$	$5\pi/4$	$-\frac{1}{\sqrt{2}} \simeq -0.71$	± 0.54
7	270	$3\pi/2$	0	± 1
8	$315 = 270 + 45$	$7\pi/8$	$\frac{1}{\sqrt{2}} \simeq 0.71$	± 1.31
9	360	2π	1	± 1.41

2. While plotting, label the positive and negative r points as $(1, 1')$, $(2, 2')$, etc. as shown in figures 5.8a-h (points 9 and $9'$ are in [a]).

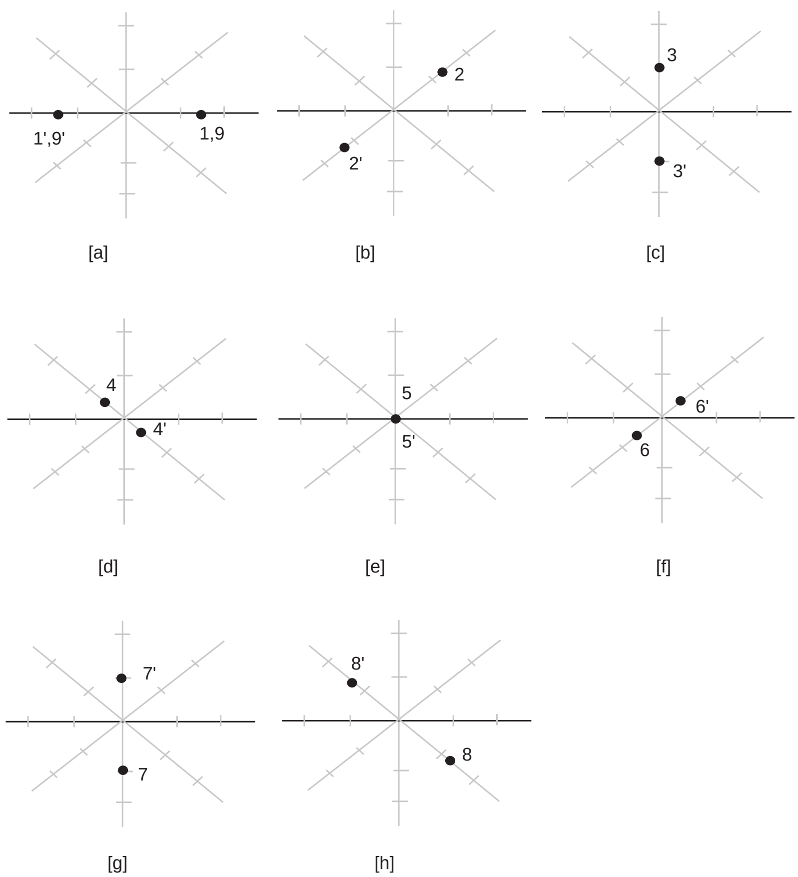


FIG. 5.8. *Plots of individual point pairs for $r^2 = 1 + \cos\theta$.*

Join the points with labels $1, 2, \ldots$ with a curve (figure 5.9 a) and the points labels $1', 2', \ldots$ with a curve (figure 5.9 b).

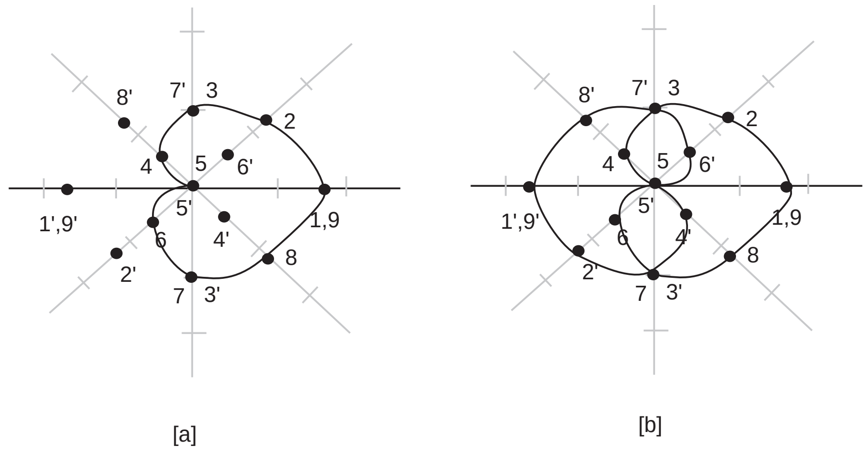


FIG. 5.9. *Complete plot of* $r^2 = 1 + \cos\theta$.

Of course if the argument contains an integer multiple of θ then you would modify that procedure with two points in each row of the table.

Finally, note that if an odd power of r is gives such as

$$r^3 = 1 + \cos\theta,$$

then you are perfectly justified in the simplification $r = (1 + \cos\theta)^{1/3}$. It is only for even powers that you have consider the positive and negative roots.

6. Final Comments. That's all there is to polar plotting. If you have read this note straight through, you may be a little overwhelmed. Just so that you don't lose your perspective, I will state the four basic ideas in this note again:

1. A coordinate system is a rule for mapping pairs of numbers to a point in the plane. Different coordinate systems have different rules.
2. You can visualize a coordinate system by its grid. The grid is drawn by fixing one coordinate and varying the other.
3. Given two coordinate systems you can convert the coordinates of a point from one system to another.
4. Given a function you can plot its graph in the coordinate system by taking the input to the function as one coordinate and the output from the function as the other coordinate.

Finally, I encourage you to do problems from your book and elsewhere. With practice, all of this will seem simple and obvious to you and you may be able to reliably omit some of the steps in the plotting.

7. Practice problems. To practice plotting with polar coordinates, use the following books in addition to your textbook. All of these books are available in the La Casa Library. My suggestion is to start with the first practice problem and keep going till you are confident that you have understood plotting with polar coordinates. At that point, you may stop if you want.

1. Problems 35.30 - 35.48 on pages 291-295 in *3000 Solved Problems in Calculus*, by Elliot Mendelson.

I suggest that you try problems 35.32, 35.38 and 35.44 on your own before you look at their solutions.

2. Problems 44 and 45 on page 383 in *Schaum's Outlines: Calculus*, by Frank Ayres and Elliott Mendelson. Just plot the curves, ignore the rest of the questions.
3. Problems 3-10,12 on page 367 in *Calculus and Analytic Geometry* by George B. Thomas. The solutions to some of the problems are given at the end of the book.