A GENTLE INTRODUCTION TO THE BASIC CONCEPTS OF SHAPE SPACE AND SHAPE STATISTICS

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1. Introduction. Shape is a prominent visual feature in many images. Unfortunately, the mathematical theory of shape and shape spaces is not very well known to image processors and engineers. My aim in this note is to introduce you to *shape spaces* and *shape statistics* with only undergraduate mathematics. You will not need anything more complicated than ordinary vector analysis and elementary calculus to understand this exposition. There is one section in this note where some topology is required, but the requirement is very elementary. We will easily navigate that section by diagrams and intuition. We have one advantage - the shape space in this note can be completely visualized in three dimensions.

At the end, I will point out certain ways of looking at this simple shape space which will motivate the advanced concepts you need for the fuller theory. I hope that you will go on to learn the real thing - shape spaces of n points in 2 and 3 (and even higher) dimensions.

This introduction is slow, so bear with me.

2. What is shape?. If two figures in a plane can be exactly superposed by translation, rotation, and scaling, then they have the same shape. To be more precise:

1. We think of the figure as a set of points in the plane.
2. We assume that the translation, rotation, and scaling acts on the entire plane, thereby mapping the plane onto itself.
3. When a figure carried along with the plane can be completely superposed (maps onto) another figure for some translation, rotation, and scaling, then the two figures have the same shape.

Thus, all circles have the same shape. All squares have the same shape. All rectangles with the same aspect ratio have the same shape. All similar triangles have the same shape.

The general theory of shape spaces is an extension of this idea - two sets in n-dimensional space have the same shape of they can be superposed using n-dimensional translation, rotation and scaling. We will not deal with the general theory here. If you are interested in pursuing the general theory, I suggest that you look at three books: "Statistical Shape Analysis" by Dryden and Mardia [1], "The Statistical Theory of Shape" by Small [2], and "Shape and Shape Theory" by Kendall et al. [3]. Dryden and Mardia's book is the easiest to begin with, but it does not go into many of the deeper geometric properties of shape spaces. Small's book is a good introduction to the geometric properties, but is not as extensive as Dryden and Mardia on the statistical theory. Finally, Kendall's book is the definitive work on the geometric aspects of shape spaces. For an engineer, it is also the hardest of three books to read. You need some familiarity with point-set and combinatorial topology to read it.

3. The shape theory of three points on a line. Let us now turn to investigate a simple case - the shape of three ordered points on a line (figure 2.1). Any ordered triple of points in the real line can be considered to be a point in $\mathcal{R}^3$. We simply take the triple of points (x_1, x_2, x_3) and write it as the column vector

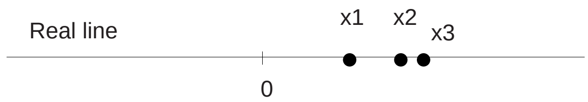
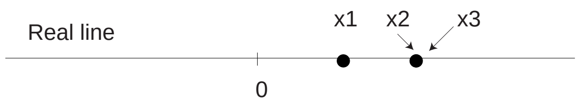


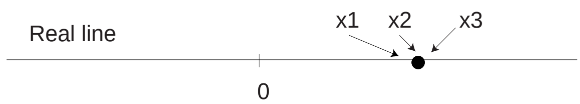
FIG. 2.1. SHAPE THEORY OF THREE POINTS IN ON THE REAL LINE. *We consider ordered subsets of the real line containing three points. Two sets have the same shape if they can be exactly mapped onto each other with translation and scaling.*

$(x_1\ x_2\ x_3)^T \in \mathcal{R}^3$. From now on, we will use the phrase "a set of three points in $\mathcal{R}$" and "a point in $\mathcal{R}^3$" interchangeably.

There is a subtle point here. Note that we considering *all* ordered triples (x_1, x_2, x_3) in the real line. These points may be all distinct ($x_1 \neq x_2 \neq x_3$) or some, or all, of them may be equal. One example of a set with two points equal is shown in figure 3.1a. If all three points are equal the set looks like figure 3.1b. The power of shape theory is that it lets us meaningfully talk about the shape of such triples.



(a) Two points equal



(b) All three points equal

FIG. 3.1. SETS WITH TWO OR MORE POINTS EQUAL.

Next, consider transformations of the real line that cause translation and scaling (there is no rotation in the real line). If $T_{\lambda,t} : \mathcal{R} \to \mathcal{R}$ is such a transformation, then,

$$T_{\lambda,t}(x) = \lambda x + t, \tag{3.1}$$

where, $\lambda \neq 0$ is the scale factor and t is the translation. We allow λ to be any non-zero number (positive as well as negative) and the translation t to be any finite real number. T is subscripted with λ and t to show that there is a different transformation T for every distinct pair of numbers λ, t.

You should immediately see that every transformation $T_{\lambda,t}$ has an inverse transformation given by ${T_{\lambda,t}}^{-1} = T_{\frac{1}{\lambda},\frac{-t}{\lambda}}$. If you don't see this, try to derive it (Hint: you need to show that if $y = T_{\lambda,t}(x)$, then $x = T_{\frac{1}{\lambda},\frac{-t}{\lambda}}(y)$).

The transformation $T_{\lambda,t}$ takes the triple points (x, y, z) to $(T_{\lambda,t}(x), T_{\lambda,t}(y), T_{\lambda,t}(z))$. That is, the transformation $T_{\lambda,t}$ induces a transformation $\mathbf{T}_{\lambda,t} : \mathcal{R}^3 \to \mathcal{R}^3$ given by

$$\begin{aligned} \mathbf{T}_{\lambda,t} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} T_{\lambda,t}(x) \\ T_{\lambda,t}(y) \\ T_{\lambda,t}(z) \end{pmatrix} \\ &= \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{3.2}$$

Again, you should check that every transformation $\mathbf{T}_{\lambda,t}$ has an inverse transformation $\mathbf{T}^{-1}_{\lambda,t} = \mathbf{T}_{\frac{1}{\lambda},\frac{-t}{\lambda}}$.

We can now define when two point sets have the same shape.

Definition: Two points $r, s \in \mathcal{R}^3$ have the same shape (as triples of points in $\mathcal{R}$) if there exists a transformation $\mathbf{T}_{\lambda,t}$ such that $r = \mathbf{T}_{\lambda,t}(s)$.

This definition is symmetric in r and s. If there exists a transformation $\mathbf{T}_{\lambda,t}$ such that $r = \mathbf{T}_{\lambda,t}(s)$, then the transformation has an inverse $\mathbf{T}^{-1}_{\lambda,t} = \mathbf{T}_{\frac{1}{\lambda},\frac{-t}{\lambda}}$ and $s = \mathbf{T}^{-1}_{\lambda,t}(r) = \mathbf{T}_{\frac{1}{\lambda},\frac{-t}{\lambda}}(r)$.

We can now ask : Given a point $(x_1\ x_2\ x_3)^T \in \mathcal{R}^3$, what is the set of all points that have the same shape as it? Applying the above definition, we see that this set contains points

$$\mathbf{T}_{\lambda,t}(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{3.3}$$

for all $\lambda \neq 0$ and all t. Let us denote it by $S_{(x_1x_2x_3)}$. This set contains all triple points that have the same shape. And any triple that is not in this set definitely has a different shape. Thus, we can say that this set *defines* the *shape* of $(x_1x_2x_3)^T$. Henceforth we will refer to $S_{(x_1x_2x_3)}$ as the shape of $(x_1x_2x_3)^T$.

To visualize $S_{(x_1x_2x_3)}$, consider two cases:

1. The vectors $(x_1\ x_2\ x_3)^T$ and $(1\ 1\ 1)^T$ are linearly independent.
2. The vectors $(x_1\ x_2\ x_3)^T$ and $(1\ 1\ 1)^T$ are linearly dependent.

3.1. Linearly independent vectors. The set $S_{(x_1x_2x_3)}$ is simply the set of points generated by equation (3.3) for all λ and all t minus the set of points generated for $\lambda = 0$ and all t. Because $(x_1\ x_2\ x_3)^T$ and $(1\ 1\ 1)^T$ are linearly independent, the first of these sets is *span*$(\ (x_1\ x_2\ x_3)^T, (1\ 1\ 1)^T)$ while the second set is *span*$(\ (1\ 1\ 1)^T)$. Thus,

$$S_{(x_1,x_2,x_3)} = \textit{span}(\ (x_1\ x_2\ x_3)^T, (1\ 1\ 1)^T) - \textit{span}(\ (1\ 1\ 1)^T). \tag{3.4}$$

NOTE: *Recall that the span of vectors $u, v, w \ldots$ is the set $span(u, v, w, ...) = \{\alpha u + \beta v + \gamma w + \ldots\}$ for all real numbers $\alpha, \beta, \gamma, \ldots$. The span of two linearly independent vectors is a plane. The span of a single non-zero vector is a line.*

Thus the shape is a plane containing ($(1\ 1\ 1)^T$) from which the line given by *span*($(1\ 1\ 1)^T$) is deleted. Figure 3.2 shows some shapes.

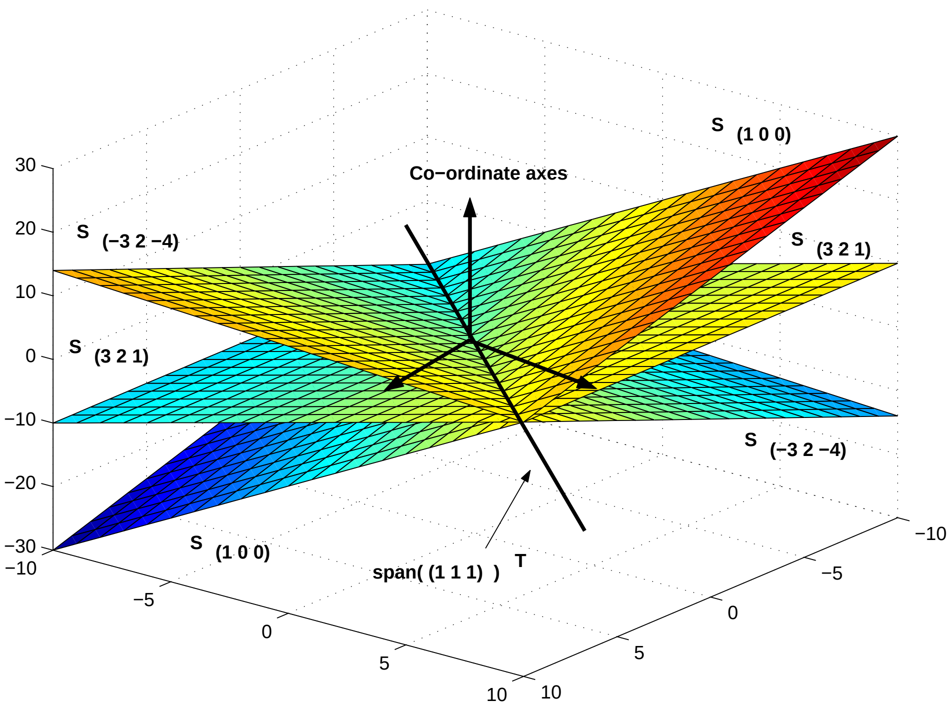


FIG. 3.2. SHAPES FOR TRIPLES $(x\ y\ z)^T$ THAT ARE LINEARLY INDEPENDENT OF $(1\ 1\ 1)^T$. *Each shape is a plane containing* $(1\ 1\ 1)^T$ *with span*($(1\ 1\ 1)^T$) *deleted.*

It is cumbersome to use the phrase "a plane containing ($(1\ 1\ 1)^T$) from which the line given by *span*($(1\ 1\ 1)^T$) is deleted" every time we refer to a shape. We will use the italicized *plane* to refer to a plane containing ($(1\ 1\ 1)^T$) from which the line given by *span*($(1\ 1\ 1)^T$) is deleted. We retain the plain "plane" for its ordinary meaning.

Figure 3.2 suggests that shapes are *plane*s that "rotate" around $(1\ 1\ 1)^T$. We can say this more precisely: Let $\mathcal{P}$ be the set of all *plane*s. Further, let $\mathcal{S}$ be the set of all shapes given by equation (3.4), then

Proposition 1: $\mathcal{S} = \mathcal{P}$.

Proof: From equation (3.4) we know that $\mathcal{S} \subset \mathcal{P}$. To establish the result we show that $\mathcal{P} \subset \mathcal{S}$. Suppose that $P \in \mathcal{P}$. Then P is a plane, say π containing $(1\ 1\ 1)^T$ minus the line *span*($(1\ 1\ 1)^T$). Since π is two dimensional and contains $(1\ 1\ 1)^T$, it must contain at least one vector, say $(x\ y\ z)^T$, that is linearly independent of $(1\ 1\ 1)^T$. Further, π itself is the span of $(x\ y\ z)^T$ and $(1\ 1\ 1)^T$. That is,

$$\begin{aligned}\pi &= \ span(\ (x\ y\ z)^T, (1\ 1\ 1)^T), \text{and,}\\ P &= \pi - \ span(\ (1\ 1\ 1)^T)\\ &= \ span(\ (x\ y\ z)^T, (1\ 1\ 1)^T) - \ span(\ (1\ 1\ 1)^T)\\ &= S_{(xyz)}.\end{aligned}$$

Hence, $P \in \mathcal{S}$ and we have shown that $\mathcal{P} \subset \mathcal{S}$.

3.2. Linearly dependent vectors. When $(x_1\ x_2\ x_3)^T$ and $(1\ 1\ 1)^T$ are linearly dependent we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence all such sets have equal three points (like figure 3.1b). Because $(x_1\ x_2\ x_3)^T$ and $(1\ 1\ 1)^T$ are linearly dependent, $S_{(x_1 x_2 x_3)} = \ span(\ (1\ 1\ 1)^T)$. That is, all such points have a single shape.

3.3. Classification. We have reached a complete characterization of shapes of three points on a line. All triple points of the form $(\alpha\ \alpha\ \alpha)^T$ have a single shape, which is *span*($(1\ 1\ 1)^T$). All other triple points have a shape which is some plane containing $(1\ 1\ 1)^T$ minus *span*($(1\ 1\ 1)^T$).

The following properties of shapes are easy to establish (I won't prove them here. The proofs are easy, you should try):

1. Shapes (as sets in $\mathcal{R}^3$) do not intersect.
2. Even though shapes do not intersect, they come arbitrarily close to each other. The intersection of the closure of shapes of $(x,\ y,\ z)^T \neq (\alpha\ \alpha\ \alpha)^T$ is *span*($(1\ 1\ 1)^T$).

Finally, let us agree that points of the type $(\alpha\ \alpha\ \alpha)^T$ are not very interesting. Such triples look just like a single point and there isn't much to say about them. From now on, we will ignore such points. All triples we consider will be linearly independent of $(1\ 1\ 1)^T$.

4. The shape space of three points on a line. Having defined the shape of three points, we proceed to define the shape space of three points:

Definition: $\mathcal{S}$, the set of all shapes, is the *shape space* of three points on a line.

Proposition 1 tells is that the shape space $\mathcal{S}$ is the set of all *planes* containing $(111)^T$. However, the shape space has additional structure. Loosely speaking, $\mathcal{P}$ (and hence $\mathcal{S}$) appears to be "smoothly connected," i.e., one can continuously "rotate" any element of $\mathcal{P}$ around $(1\ 1\ 1)^T$ to get any other element.

Our next task is to make this intuition more precise. To do this in the classical definition-proof style of mathematics we need tools from elementary topology which we do not have. So there will be no definitions or proofs in the next section. We will rely on figures and intuition to make the argument.

5. The topology of shape space. Let σ be the plane through the origin perpendicular to $(1\ 1\ 1)^T$. Every point in $\mathcal{R}^3$ has a unique projection on σ. The projection is given by the projection operator $\pi_\sigma : \mathcal{R}^3 \to \sigma$ given by

$$\begin{aligned}\pi_\sigma(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{1}{3}\left\{(1\ 1\ 1)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right\}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x_1 - \frac{x_1+x_2+x_3}{3} \\ x_2 - \frac{x_1+x_2+x_3}{3} \\ x_3 - \frac{x_1+x_2+x_3}{3} \end{pmatrix}. \end{aligned} \tag{5.1}$$

The projection operator is a linear operator.

Let $(x_1\ x_2\ x_3)^T$ be linearly independent of $(1\ 1\ 1)^T$ and let $S_{(x_1 x_2 x_3)}$ be its shape. Then, according to equation (3.3) the elements of $S_{(x_1 x_2 x_3)}$ are given by

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{5.2}$$

for $\lambda \neq 0$. The projection of $S_{(x_1 x_2 x_3)}$ onto the plane σ is the set $\pi_\sigma(S_{(x_1 x_2 x_3)})$ whose points are given by

$$\begin{aligned}\pi_\sigma(\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) &= \pi_\sigma(\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) + \pi_\sigma(t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) \\ &= \pi_\sigma(\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) \\ &= \lambda \pi_\sigma(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}),\end{aligned}$$

for $\lambda \neq 0$. Since $(x_1\ x_2\ x_3)^T$ is linearly independent of $(1\ 1\ 1)^T$, its projection $\pi_\sigma((x_1\ x_2\ x_3)^T)$ is not zero. Therefore, according to the above equation, the projection $\pi_\sigma(S_{(x_1 x_2 x_3)})$ is a line (in σ) through the origin in the direction $\pi_\sigma((x_1,\ x_2,\ x_3)^T$ with the origin deleted.

It is now straightforward to show that every shape projects onto a unique line through the origin of σ with the origin deleted. As we did for planes, we will adopt the italicized *line* to denote line though the origin with the origin deleted.

Our first claim is that *line*s through the origin are in one-to-one correspondence with shapes (you should try to prove this. It only requires vector analysis). This is illustrated in figure 5.1.

Let S^1 be a unit circle centered at the origin of σ (fig. 5.1). Then the projection of every shape, which is a *line*, intersects S^1 in two diametrically opposite points. The set of *line*s is clearly in one-to-one correspondence with pairs of diametrically opposite points on S^1.

Finally, let us parameterize the points on S^1 by their angle $\theta \in [0, 2\pi)$ and consider the following function $f : S^1 \to S^1 \times S^1$

$$f(\theta) = (\theta/2, \theta/2 + \pi), \tag{5.3}$$

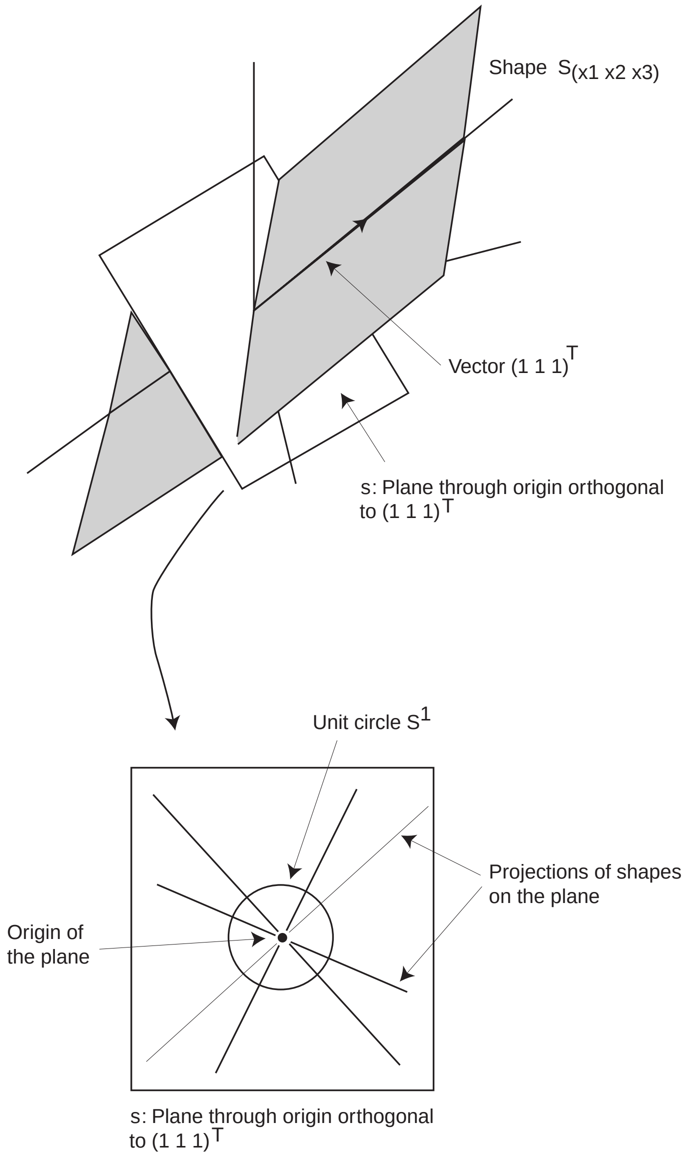


Fig. 5.1. Projection of shape onto the plane σ.

which takes the point θ of S^1 to the pair of diametrically opposite points $(\theta/2, \theta/2+\pi)$ on the unit circle (figure 5.2). As a function from points on the circle to pairs of diametrically opposite points of the circle, the function is one-to-one. This shows that diametrically opposite points of a circle are in one to one correspondence with points on a circle.

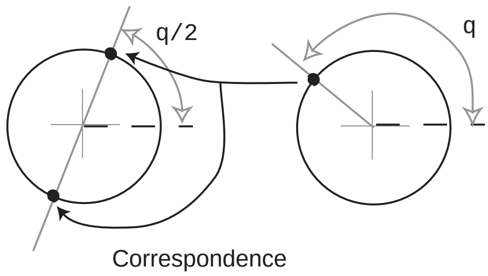


FIG. 5.2. PAIRS OF DIAMETRICALLY OPPOSITE POINTS OF A CIRCLE ARE IN ONE-TO-ONE CORRESPONDENCE WITH POINTS ON A CIRCLE.

We now have long chain of one-to-one correspondences

$$\begin{gathered}\text{Shapes of three points on the one real line}\\ \updownarrow \\ \text{Lines through the origin of } \sigma \text{ without the origin}\\ \updownarrow \\ \text{Pairs of diametrically opposite points of } S^1 \\ \updownarrow \\ \text{Points of } S^1\end{gathered}$$

Following this chain, we see that shapes are in one-to-one correspondence with points of a circle.

You can think of each one-to-one correspondence as a one-to-one and onto function from one set to the next. Because the correspondence is one-to-one, the function has an inverse. Also, you should be able to see intuitively that all of the correspondence functions and their inverses are continuous.

If there is a one-to-one and onto function $f : X \to Y$ between any two sets X and Y such that the function and its inverse are continuous, then the two sets are topologically identical. Loosely speaking, this means that you can "rubber sheet" one set onto the other without tearing the sheet. The mathematical terminology for this is to say that X and Y are *homeomorphic*. Thus, we say

Proposition 2: The shape space of three points on the real line is homeomorphic to the plane circle.

We can go even further. It is possible to show that the shape space is not just in one-to-one correspondence with a circle, but the correspondence is of a type that allows shape space to inherit calculus (the capacity to differentiate functions) from the circle. This make shape space a *differential manifold.*

6. The Procrustes Distance. We can define a meaningful distance in the shape space - the so called *Procrustes* distance. To define it we need some explicit formulae for all the one-to-one correspondences of the previous section. Some of these formulae we have seen before. They are reproduced here for convenience.

To recall the correspondences of the previous section, suppose $u = (x\ y\ z)^T \in \mathcal{R}^3$ is point that is linearly independent of $(1\ 1\ 1)^T$. Its shape is S_u and the projection of S_u onto the plane σ is the set

$$\lambda \begin{pmatrix} x_1 - \frac{x_1+x_2+x_3}{3} \\ x_2 - \frac{x_1+x_2+x_3}{3} \\ x_3 - \frac{x_1+x_2+x_3}{3} \end{pmatrix}, \text{ for } \lambda \neq 0.$$

Let

$$\tilde{u} = \begin{pmatrix} x_1 - \frac{x_1+x_2+x_3}{3} \\ x_2 - \frac{x_1+x_2+x_3}{3} \\ x_3 - \frac{x_1+x_2+x_3}{3} \end{pmatrix}, \tag{6.1}$$

then, $\tilde{u} \neq 0$, and the projection of S_u on σ is $\lambda\tilde{u}$, for $\lambda \neq 0$. The projection intersects the unit circle at the two diametrically opposite points $\pm u^*$, where

$$u^* = \frac{\tilde{u}}{\parallel \tilde{u} \parallel} \tag{6.2}$$

and $\parallel \tilde{u} \parallel$ is the Euclidean norm of $\tilde{u}$.

Recall from above that we defined the map from S^1 to the pair of diametrically opposite points in terms of an angular parameterization of the circle. Let a be some unit vector with respect to which we are willing measure angles, and let θ_{u^*} be the angle from a to u^* or $-u^*$, whichever occurs first going counter-clockwise from a. Then, according to equation (5.3), the diametrically opposite points $(\theta_{u^*}, \theta_{u^*} + \pi)$ are represented by a single point $f^{-1}((\theta_{u^*}, \theta_{u^*} + \pi)) = 2\theta_{u^*}$ on S^1. Thus shape of $u \in \mathcal{R}^3$ can be represented as the point $2\theta_{u^*}$ on the unit circle. This representation does require us to choose the axis a from which we measure angles.

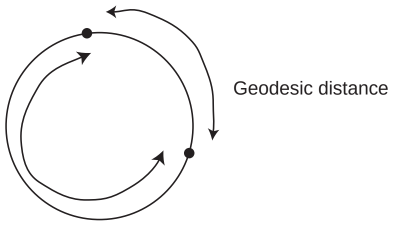


Fig. 6.1. The geodesic distance on the circle.

There is a natural distance between any two points on the circle, which is simply the smallest of the two possible circumferential distances from the first point to the second (figure 6.1). This is the *geodesic distance* on the circle. We will denote the geodesic distance between the point at θ_1 and the point at θ_2 by $d_{S^1}(\theta_1, \theta_2)$. The geodesic distance is independent of axis from which the angles of the two points are measured

We can define a distance between the shapes of $u, v \in \mathcal{R}^3$ as any number proportional to $d_{S^1}(2\theta_{u^*}, 2\theta_{v^*})$ with a positive constant of proportionality. We will shortly see that a constant of proportionality of $\frac{1}{2}$ is geometrically meaningful. So,

Definition: The *Procrustes distance* between the shapes of two sets u, v of triple points on a line is

$$\rho(u, v) = \frac{1}{2} d_{S^1}(2\theta_{u^*}, 2\theta_{v^*}).$$

The Procrustes distance measures the distance between the shapes of u and v. The reason for choosing a constant of proportionality equal to 1/2 is that the Procrustes distance turns out to be the minimum geodesic distance between the diametrically opposite pairs of points $\pm u^*$ and $\pm v^*$.

7. Shape Statistics. Now that we understand shape spaces, we can talk about shape probabilities and statistics. Some general remarks first: The problem of shape statistics is to summarize (and use) the shape statistics of large collections of triples on $\mathcal{R}^3$. Usually this requires us to define parameterized probability densities of shapes and we estimate the appropriate parameter values given the data set.

In Euclidean spaces this is all rather standard. A typical multidimensional probability density is the Gaussian (or the Normal distribution) which has the mean and correlation matrix as its parameters. In any application where this density is used, these parameters are estimated from the data set.

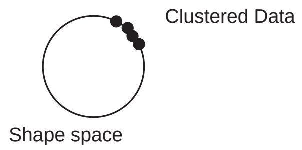
The idea is the same for our shape space, with one major difference - our shape space is a circle. Thus probability distributions of shape are distributions on a circle. They are quite well-known in statistics [4], and can be easily used in our shape space.

Typically, one processes a data set as follows. Every triple $u = (x\ y\ z)^T$ in the data set is transformed via equations (6.1-6.2) into its shape u^*. The set of shapes obtained this way are used to estimate the parameters of some probability distribution on the circle. The distribution with the estimated parameters is then used in subsequent processing such as hypothesis testing.

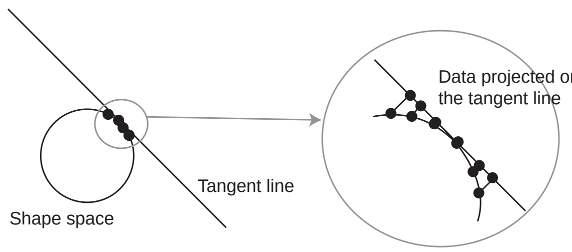
Because our shape space is a circle, statistical calculations (such as parameter estimation) can become more complicated than in an ordinary Euclidean space. Fortunately, it is often possible to approximate shape statistics with Euclidean statistics. The key observation that makes this possible is this: *in practice most shape data sets are tightly clustered in the shape space.* When you think about it, this is not that surprising. Most data sets are collections of objects of a single type, such as images of a specific organ, say the spine, and the shape variation within the same type of object tends to be small.

When the distribution of shape in a data set is tight, the shape space can be approximated by its tangent line as shown in figure 7.1. The a part of the figure shows our shape space (the circle) and a set of tightly clustered shapes in the shape space. If we choose a point on the circle that is near the center of the clustered shapes (say at the mean), then we can construct a tangent line to the circle at that point (fig. 7.1b). Further, if we project every data point from the circle to the tangent line, then we can do ordinary Euclidean statistics on the projected data points. The difference between these statistics and the true statistics on the circle can be really small if the data set is clustered tightly around its mean. For many data sets, the difference is negligible and all of classical statistical machinery of Euclidean statistics can be used readily with the projected data.

To summarize - shape statistics are often created as follows: First all data $u = (x\ y\ z)^T$ is projected on the shape space. A convenient "center" for the data is found and the shape space is approximated by its tangent line at the center. The data are



(a) Clustered Data in the Shape space



(b) Data projected onto the tangent line

FIG. 7.1. APPROXIMATING SHAPE STATISTICS WITH EUCLIDEAN STATISTICS. *Typically many data sets are tightly clustered in shape space. These data can be projected onto an appropriate tangent line and Euclidean statistics can be used on the projected data.*

further projected onto the tangent line and ordinary Euclidean statistics are used with the projected data.

This procedure is also used in higher dimensional shape spaces, where the data are projected onto an appropriate tangent plane passing through a center point. As you can guess, there are many different ways of choosing the center point, and even different ways of projecting the data points onto the tangent plane. They are discussed in detail in Dryden and Mardia [1]. I recommend that you look there for further details.

8. Comments. We have now completed our discussion of the shape spaces and shape statistics of three points on a line. We saw that (excluding the case where all three points were coincident) the shape space is a circle and the natural geodesic distance on the circle is one appropriate metric for the shape space. Further statistical analysis of shape can often be carried out on an appropriately chosen tangent line.

The general theory of shape spaces proceeds in a similar fashion. We start with p points in $\mathcal{R}^n$. We allow a set of transformations to act on $\mathcal{R}^n$. The set of all p points which can be mapped onto each other by using one of the transformation defines a shape. The set of all such shapes is a shape space. It turns out that shape spaces are manifolds (abstract surfaces) and except for certain singularities, they are differentiable manifolds. Also as above, Euclidean shape statistics can often be used if the shape data are tightly clustered and can be projected on an appropriate tangent plane.

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