Affine Connections in Plain English

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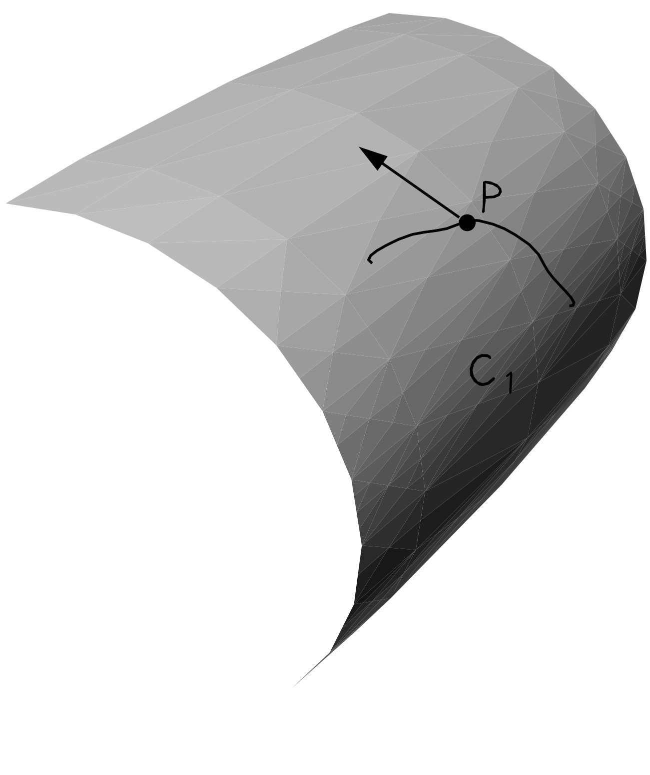
Tangent Vector and Space

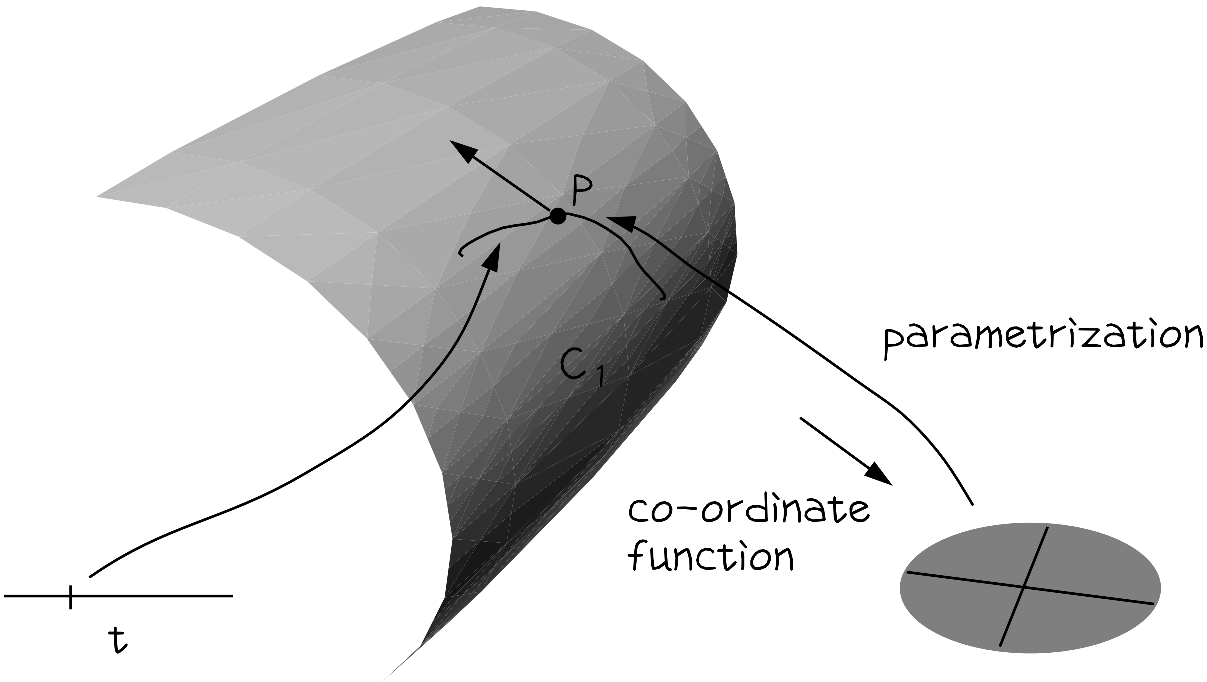
Recall:

A tangent vector is a directional derivative operator
at a point.

The set of all directional derivative operators at a point
is a vector-space and is called the tangent space
at the point.



Notation



The notation is a little funky:

We will denote tangent vectors as v_P etc.

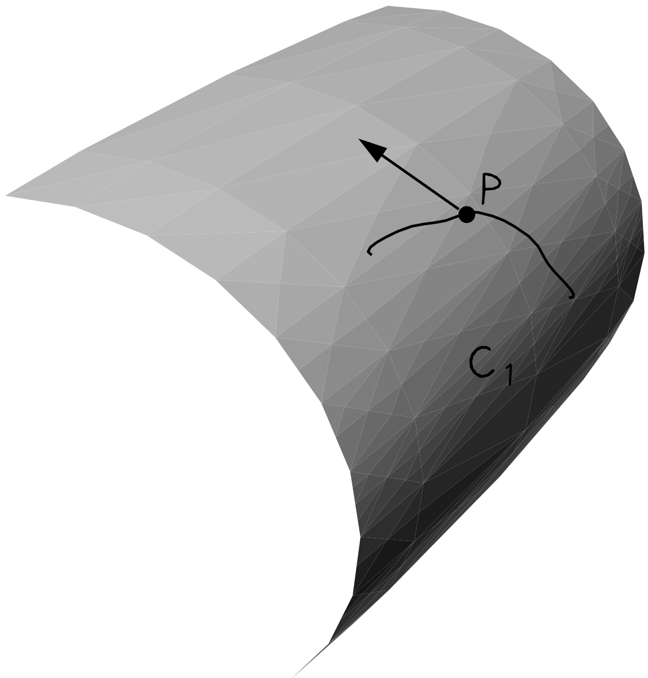
The directional dervivative of a function is denoted v [f]

Given a parametrization, the co-ordinate curves form a basis of the tangent space. This is expressed as

$$v_P = \sum_i a_i \left(\frac{d}{dx_i}\right)_P \qquad v_P[f] = \sum_i a_i \left(\frac{d}{dx_i}\right)_P [f]$$

The tangent space is denoted $T_P M$, where M is the manifold.

The Tangent Bundle



The set of all tangent spaces of a manifold is the tangent bundle of the manifold, denoted TM.

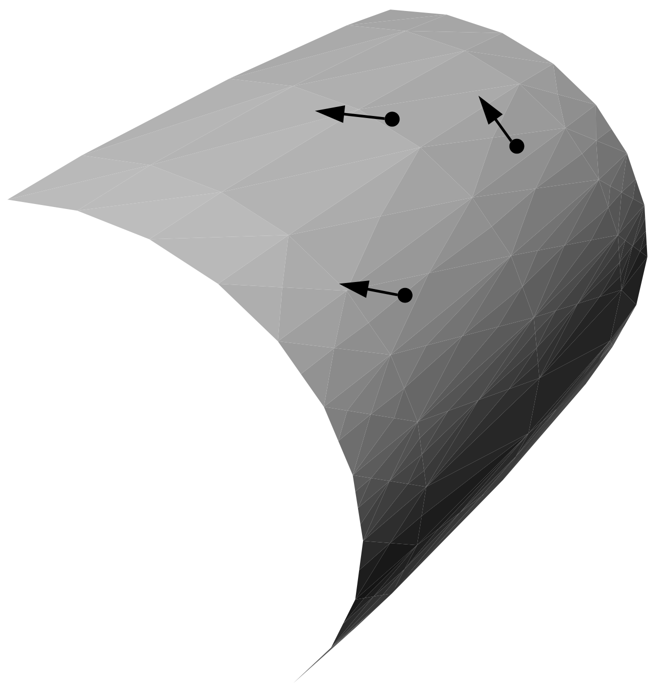
The tangent bundle is itself a manifold:

If $\{ (U_\alpha, \phi_\alpha) \}$ is an atlas for M, then

$\{(U_\alpha \times R^n, (\phi_\alpha, \gamma_\alpha))\}$ is an atlas for TM:

where, γ_α takes R^n into the tangent space at p.

Vector Field



A vector field on a manifold M is a correspondence (function) which associates with each point p of M a vector $x(p)$ of $T_p M$.

The vector field is x, its value at a point is $x(p)$.

A vector field x is a map from M to TM. $x: M \rightarrow TM$.

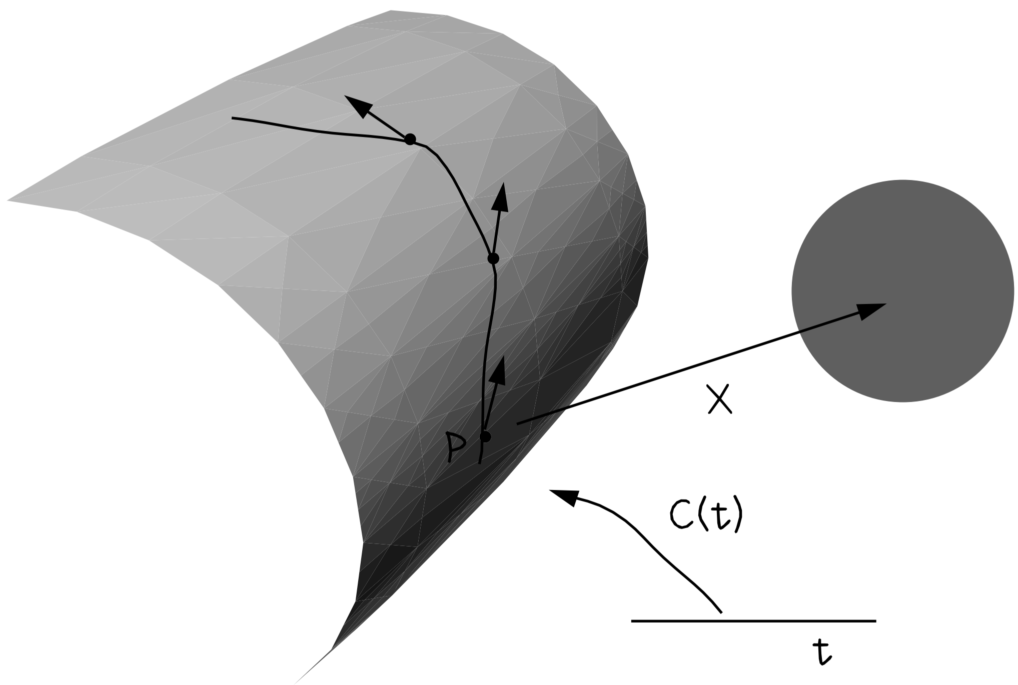
The vector field is differentiable if the map $x: M \rightarrow TM$ is differentiable

Theorem: Let a vector field v be expressed in local co-ordinates

as $v(p) = \sum_i a_i(p) \left(\frac{d}{dx_i}\right)_p$, then v is differentiable if and

only if $a_i(p)$ are differentiable functions on the manifold.

Integral curve of a vector field



<u>Defn</u>: Let v be a vector field on manifold M. An integral curve of v passing through point P of M is a curve t->C(t) such that

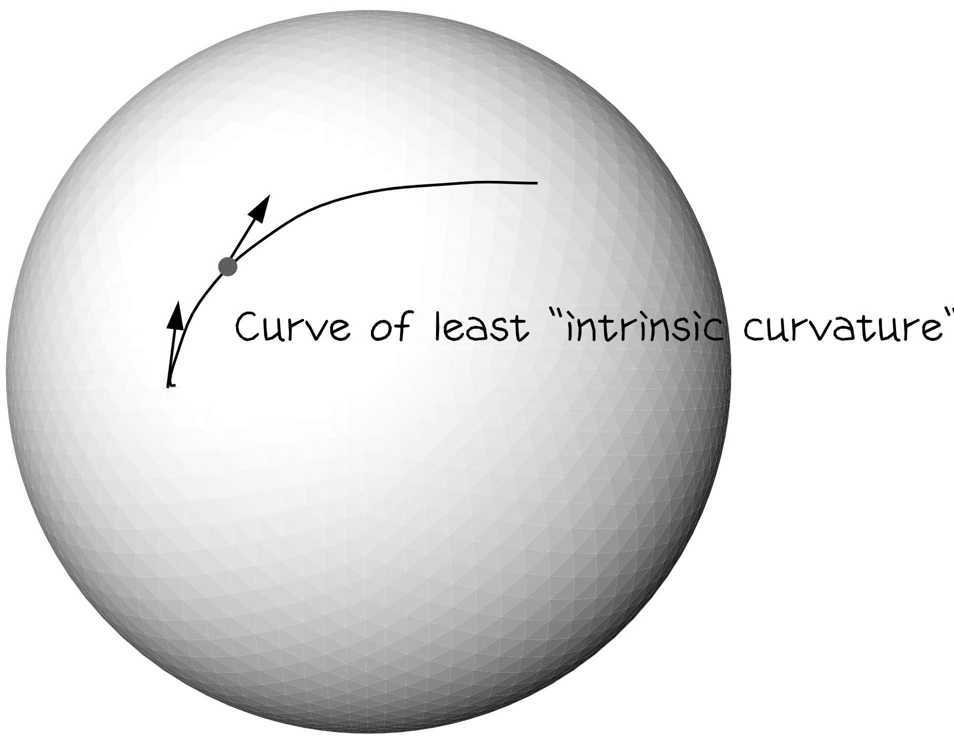
[1] C(0) = P, and [2] $\dfrac{d\,C(t)}{d\,t} = v(C(t))$.

$$\frac{d}{dt} X_i(C(t)) = a_i(C(t))$$

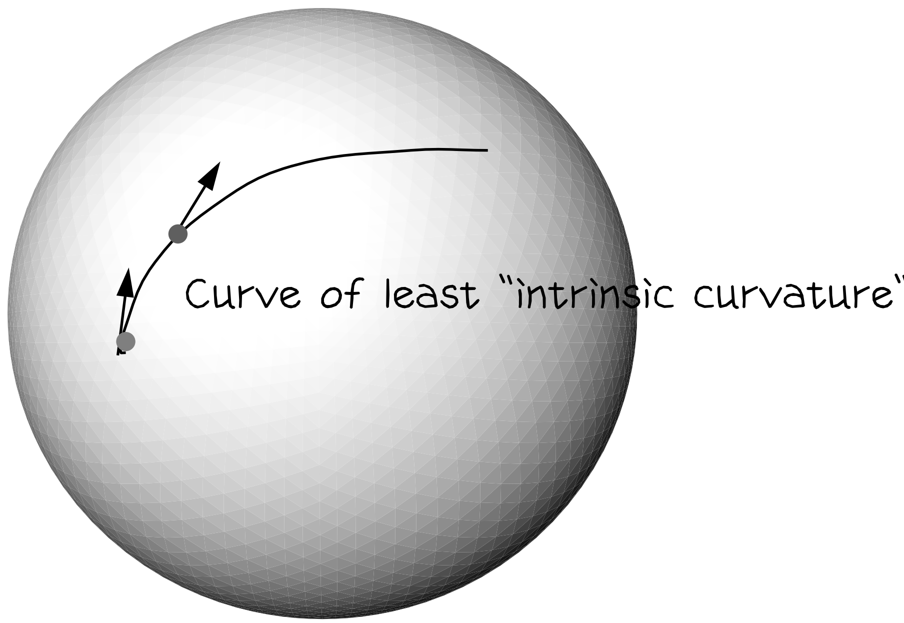
Geodesic, Parallalism, Connection

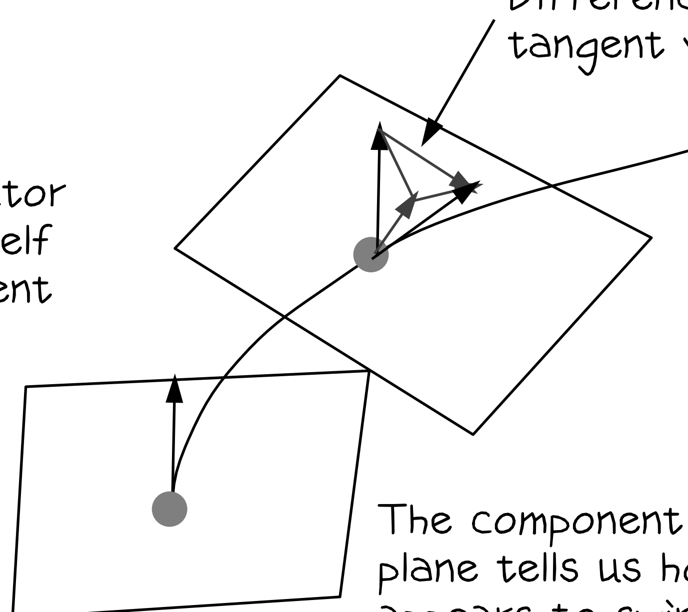
After Riemman, the next biggest conceptual leap was by Levi-Civita (1917) who formalized the notion of parallel transport, connection and hence that of a geodesic.

A clear definition of these quantities was not available even though mathematicians as far back as Euler understood what a geodesic was.



Geodesic

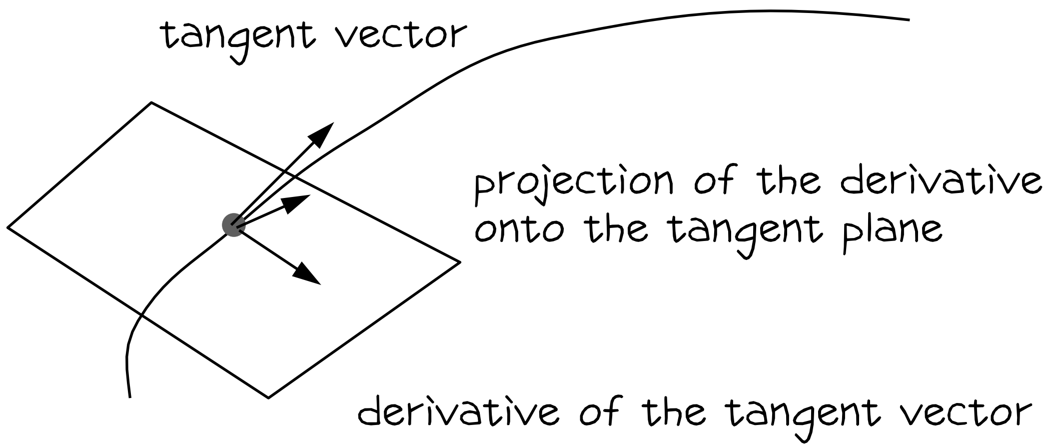




The component in the tangent plane tells us how much the curve appears to swing in the surface.

This is the notion of "intrinsic curvature"

Do this construction differentially



A geodesic is a curve for which the projection of the the derivative of the tangent vector onto the tangent plane is zero at all points of the curve.

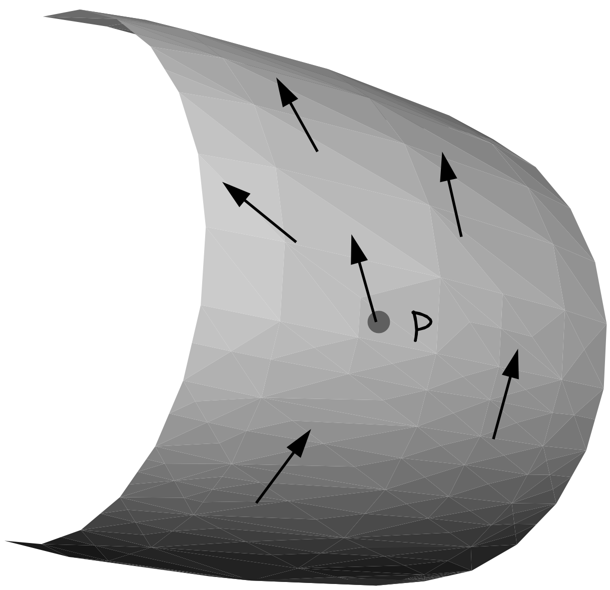
<u>Theorem</u>: Locally, a geodesic minimizes arc-length.

Note: The derivative of a tangent vector along a curve still requires the ambient space.

Projection of derivative of a tangent vector

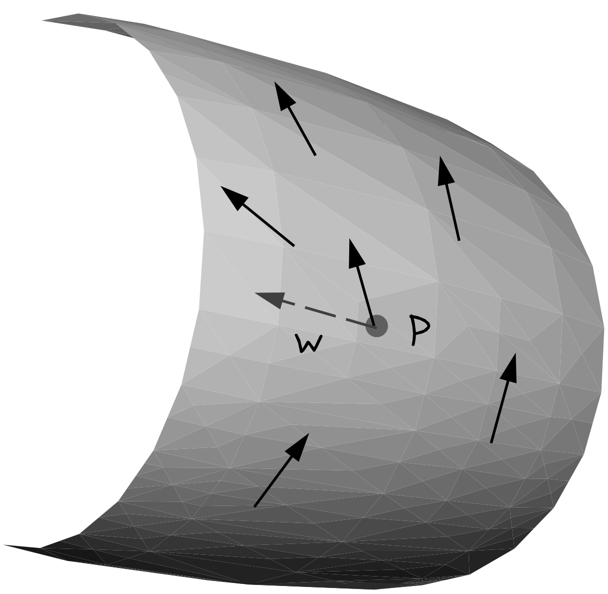
(Surface in 3-D)

[1]



Vector field v

[2]

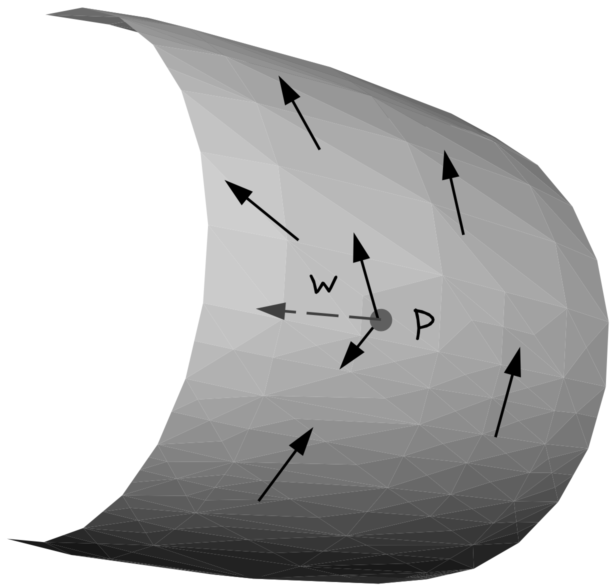


Choose a direction w
also in the tangent
space at P

Projection of derivative of a tangent vector

(Surface in 3-D)

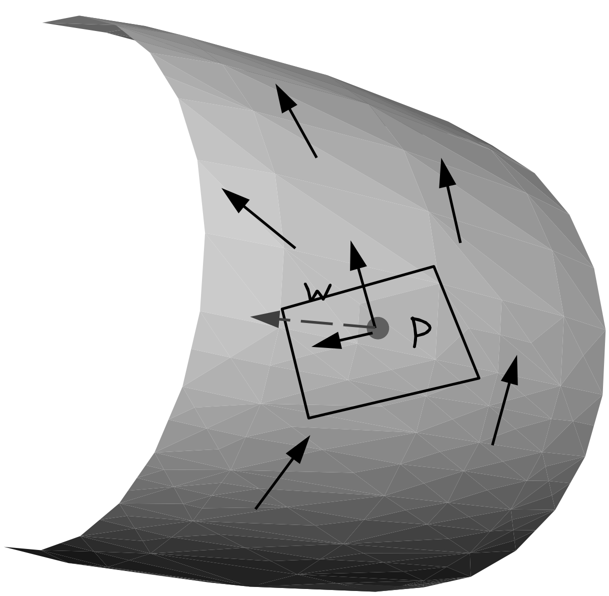
[3]



Take the derivative of v along w at P

$$\left.\frac{dv}{dw}\right|_P$$

[4]



Project it on the tangent plane at P

$$\Pi_P \left.\frac{dv}{dw}\right|_P$$

$$D_{w,P}(v) = \Pi_P \left.\frac{d}{dw}\right|_P (v)$$

$D_{w,P}(v)$ is called the <u>covariant derivative</u> of v along w at P

Covariant Derivative

By using properties of ordinary derivatives we can show that

[1] $D_{w,p}(ax + by) = a D_{w,p}(x) + b D_{w,p}(y)$,

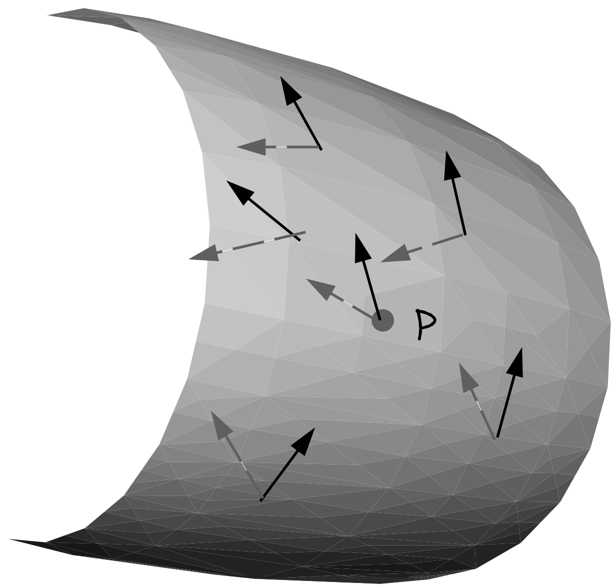
[2] $D_{av+bu,p}(x) = a D_{v,p}(x) + b D_{u,p}(x)$

Parallel Transport

Defn: A vector is parallely transported along a curve if its covariant derivative is zero.

A geodesic parallely transports its tangent vector

Affine Connection



Let w be another vector field and define ∇ to be the the operator that gives the covariant derivative of v wrt w at every point on the surface

∇ :TS X TS -> TS given by $\nabla_w(v)$ at P $= D_{w,P}(v)$

∇ is called the <u>affine connection</u>. It tells us how the tangent space at every point is "connected" to the tangent spaces around the point

For a surface it is derived via ambient space.

Affine Connection

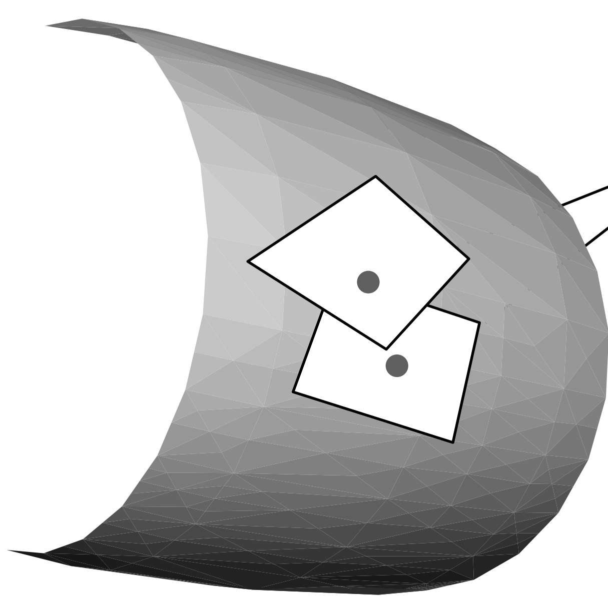
The laws of ordinary derivative give us the following properties for the affine connection

[1] $\nabla_{fv+gw}(x) = f\nabla_{v}(x) + g\nabla_{w}(x)$

[2] $\nabla_{v}(x+y) = \nabla_{v}(x) + \nabla_{v}(y)$

[3] $\nabla_{v}(fx) = f\nabla_{v}(x) + v[f]\,x$

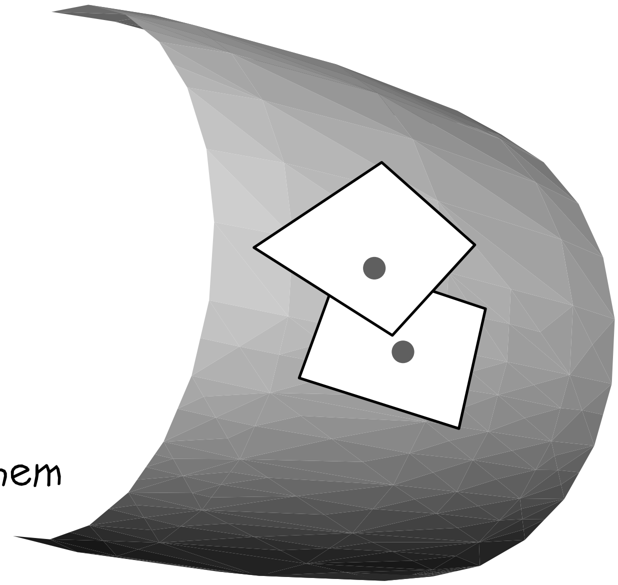
How do we derive the affine connection for a manifold ??



How are these "connected"?

i.e. how should we define a covariant derivative (a "projected" derivative)?

Affine Connection on a Manifold



The definition of a manifold does not tell us anything about how tangent spaces should be connected

That we are free to "connect" them in any way.

Choosing a "connection" is equivalent to choosing an affine connection operator.

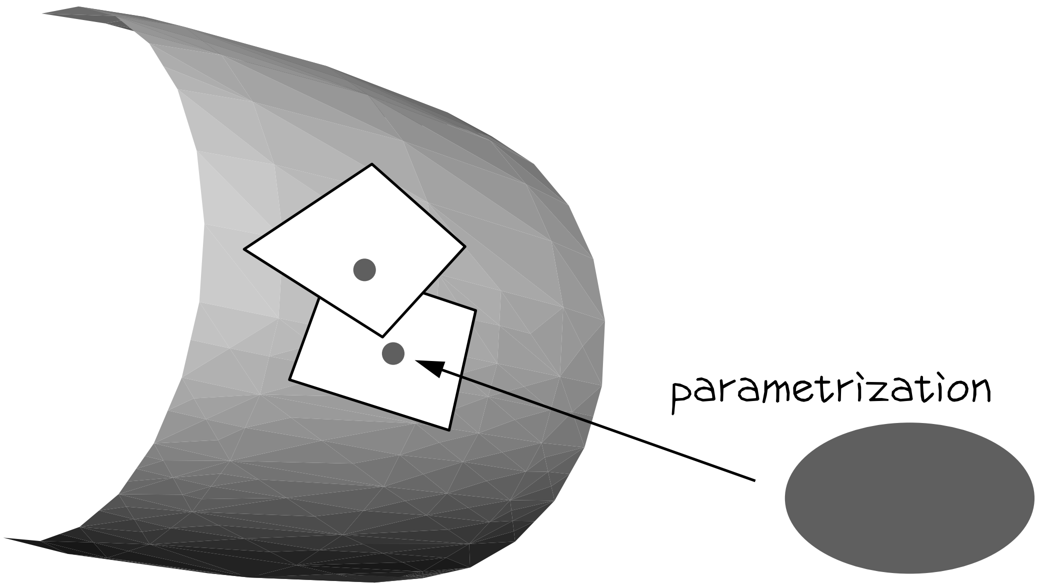
Choose any $\nabla : TM \times TM \rightarrow TM$ as long as it satisfies the formal properties [1] – [3]

[1] $\nabla_{fv+gw}(x) = f\nabla_v(x) + g\nabla_w(x)$

[2] $\nabla_v(x+y) = \nabla_v(x) + \nabla_v(y)$

[3] $\nabla_v(fx) = f\nabla_v(x) + v[f]\,x$

Affine Connection on a Manifold



Theorem:

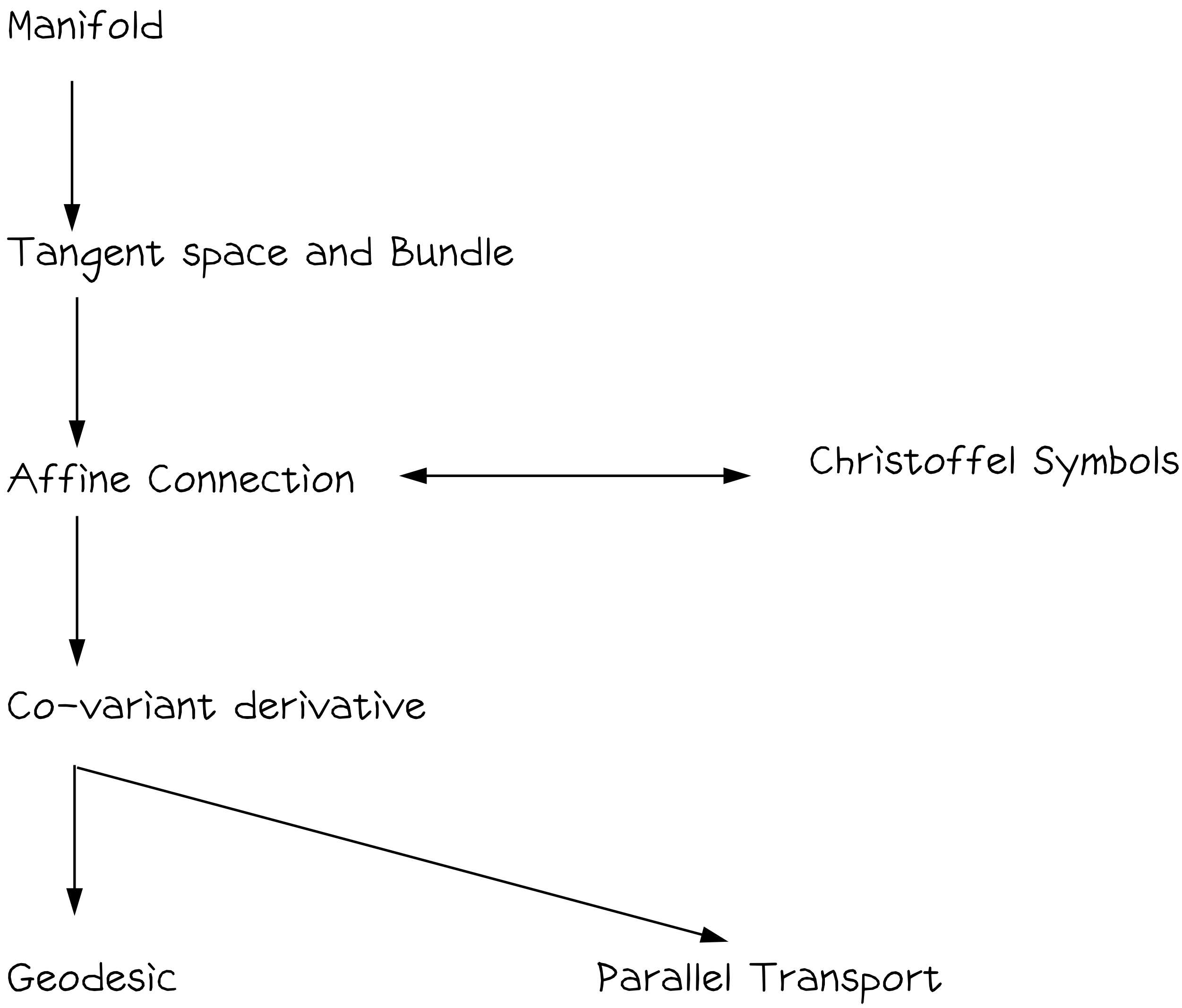
Let $x(p) = \sum_i a_i(p)\left(\frac{d}{dx_i}\right)_p$ and $y(p) = \sum_i b_i(p)\left(\frac{d}{dx_i}\right)_p$

Then,

$$\nabla_y(x) = \sum_k \left(\sum_{i,j} a_i(p)\, b_j(p)\, \Gamma^k_{i,j}(p) \quad + x[b(p)] \right)\left(\frac{d}{dx_k}\right)_p$$

Christoffel symbols (n^3). Choose any functions.
Choosing these corresponds to choosing the kth component of the co-variant derivative of the ith basis w.r.t. jth basis

Affine Connection on a Manifold



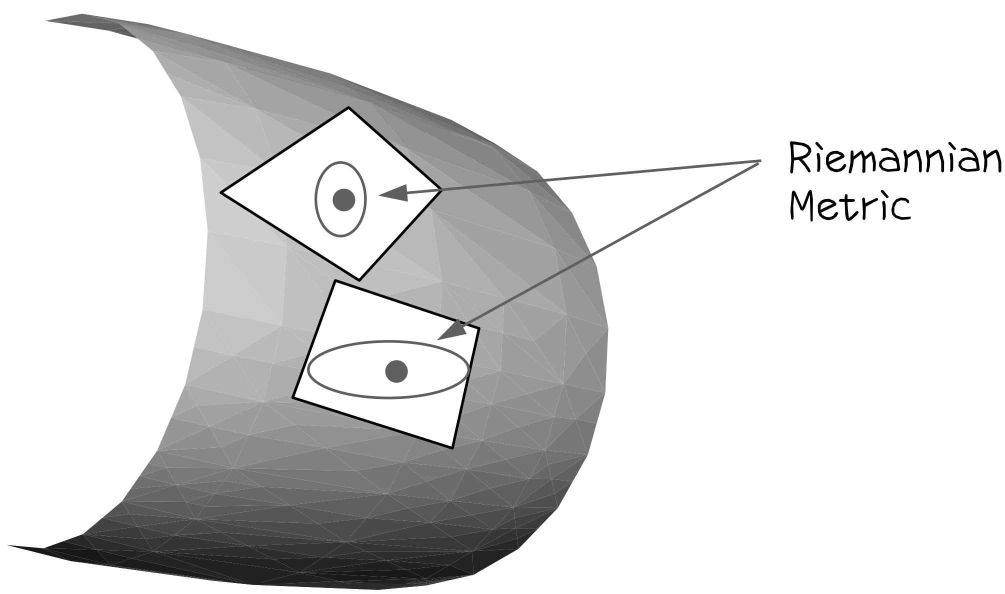
Is there a meaningful and generally useful affine connection?

Surface in 3-D : Affine connection from ambient space
Manifolds : Affine connection from Riemannian Metric

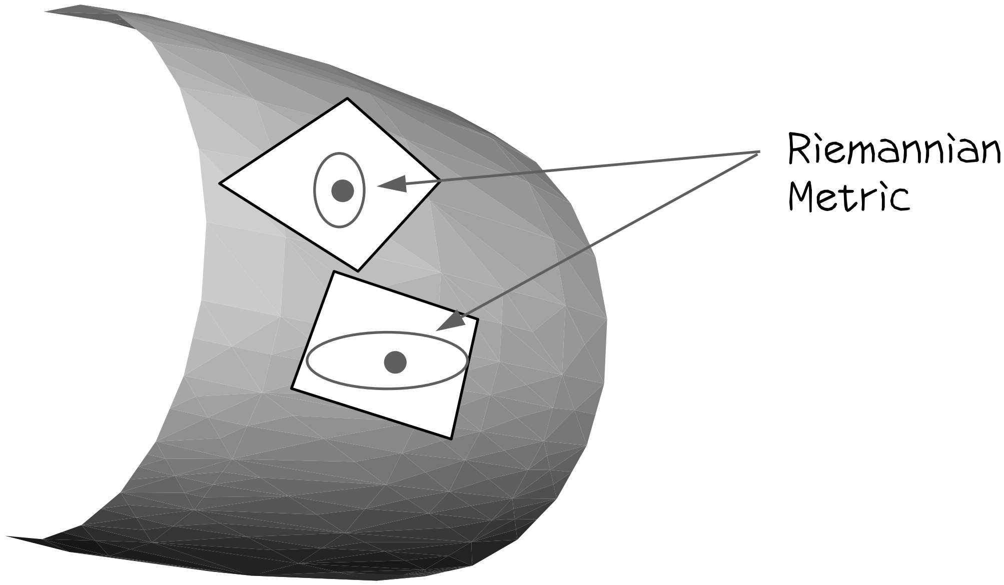
Riemannian Metric

<u>Defn:</u> A Riemmanian Metric on a manifold M is an inner product $\langle \; , \; \rangle_p$ defined for the tangent space at every point p of the Manifold.

Inner product $\langle \; , \; \rangle_p$ is bilinear, symmetric and positive definite.



Riemannian Metric

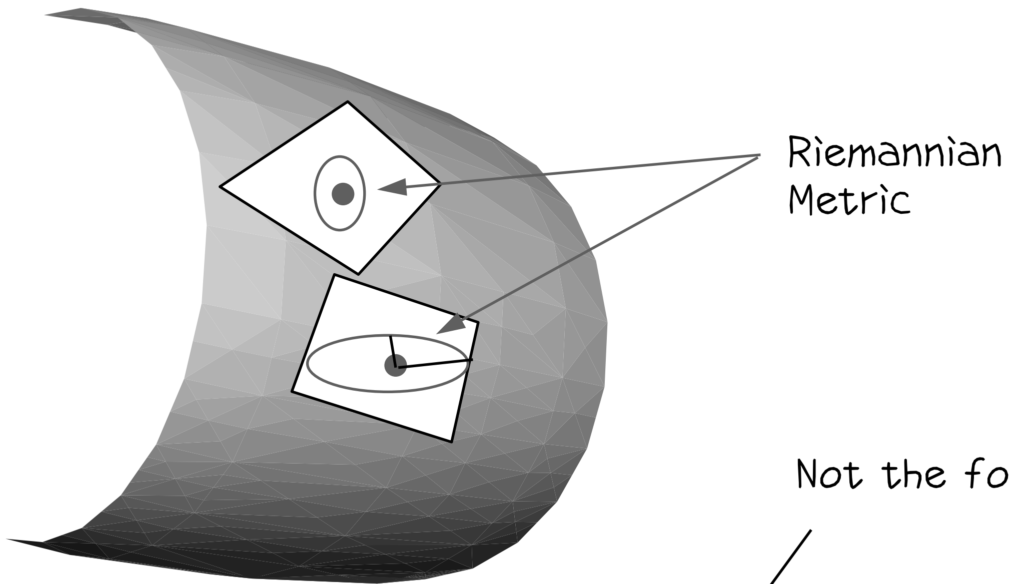


Let $x(p) = \sum_i a_i(p) \left(\frac{d}{dx_i} \right)_p$ and $y(p) = \sum_i b_i(p) \left(\frac{d}{dx_i} \right)_p$

Then, $\langle x(p), y(p) \rangle_p = \sum_{i,j} g_{i,j}(p) \; a_i(p) \; b_j(p)$

These four functions on the manifold define the Riemannian Metric.

Affine Connection and Riemannian Metric



Not the formal defn.

Is there an affine connection such that the "orthogonal frame" of the Riemannian Metric is parallel transported along itself?

Theorem (Levi-Civita): Yes!

Proof:

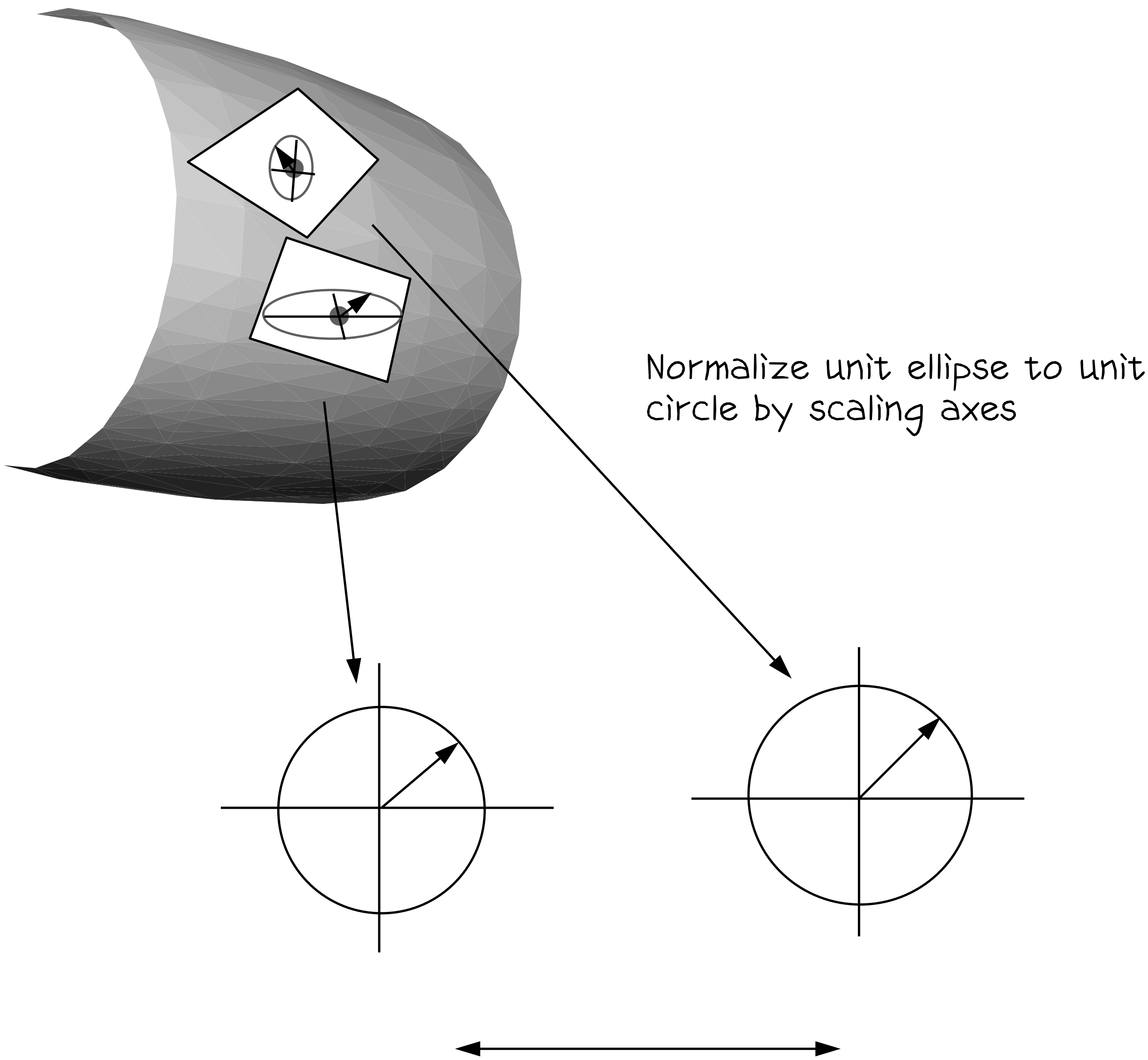
Riemannian Connection

$$\Gamma^{k}_{i,j} = \frac{1}{2} \sum_{m} \left\{ \frac{d}{dx_i} g_{j,m} + \frac{d}{dx_j} g_{m,i} - \frac{d}{dx_m} g_{i,j} \right\} (G^{-1})_{m,k}$$

where G = matrix $g_{i,j}$

Parallel Transport with a Riemannian Connection

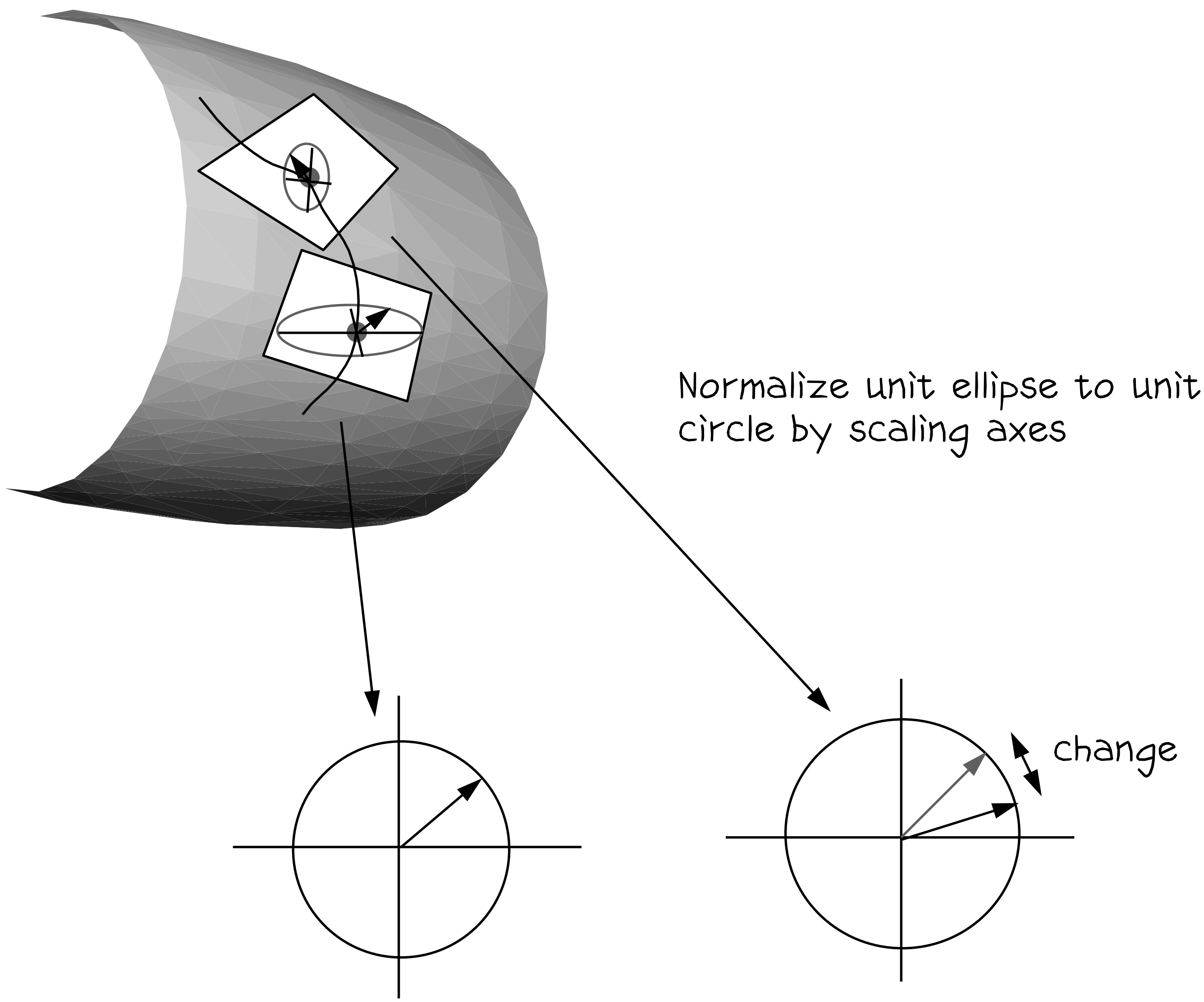
(Intuitive)



Two vectors are parallel if they have the same co-ordinates after normalization.

Covariant Derivative

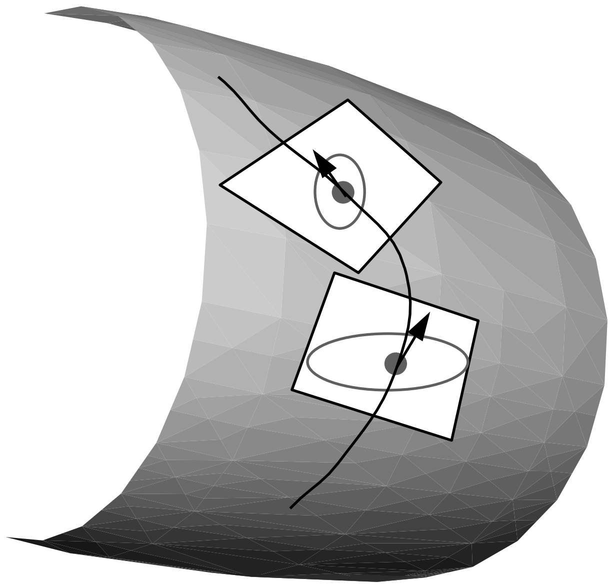
(Intuitive)



$\dfrac{d\ \text{change}}{dt}$ is the co-variant derivative

Geodesic

(Intuitive)

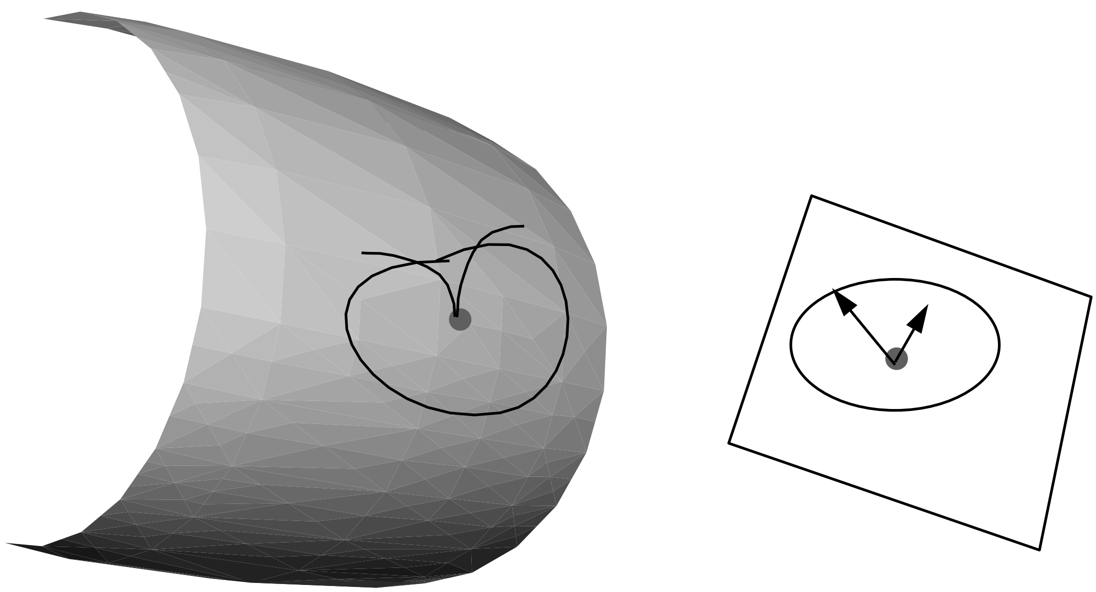


Tangent vector is transported parallel to itself.

Theorem: A geodesic minimizes local Riemannian arc-length

Geodesic

(Intuitive)



Theorem: (Intuitive)

[1] For every tangent vector in some open
ball of size (metric) e > 0, there is a corresponding unique geodesic
in the manifold.

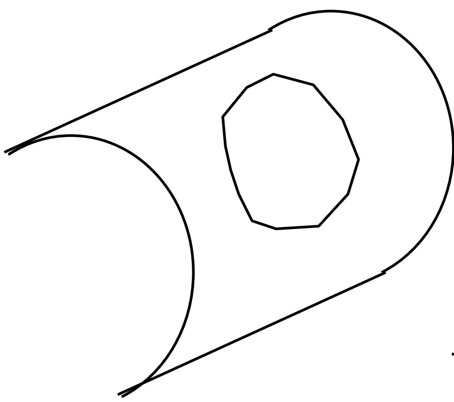
[2] Let $C_v(t)$ be the geodesic whose initial vector is v. Then

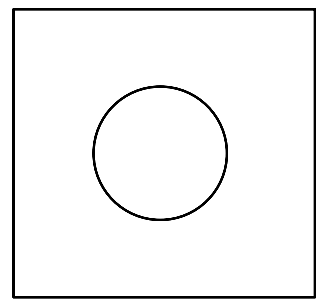
the map $\exp(v) = C_v(1)$ takes v to point in M.

$\exp(v)$ is a diffeomorphism from the open ball in the tagent space
to an open set of the manifold.

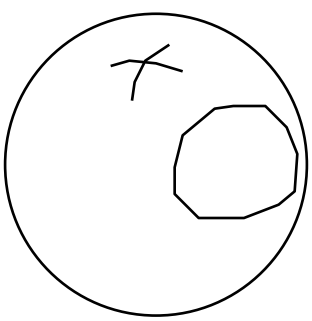
The Exponential Map

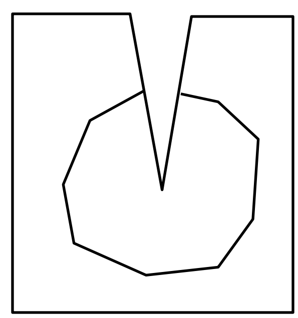
The exponential map is incredibly important in understanding the "curvature" of the space because it lets us define a "sphere" on the manifold that has the same dimension as the manifold.



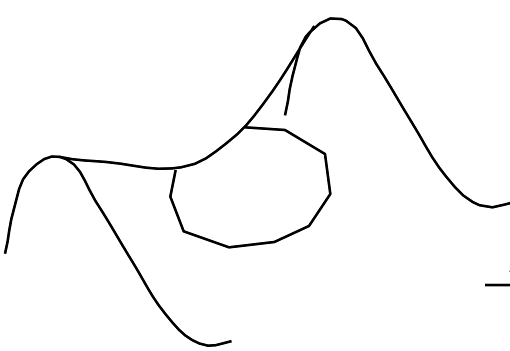


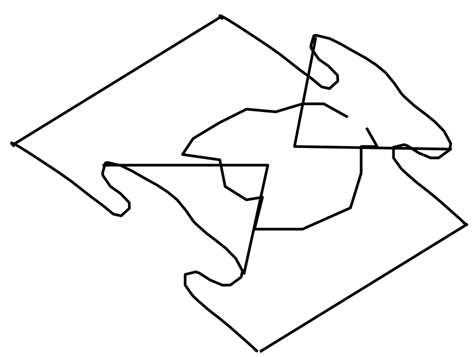
$$\frac{\text{Area}}{\text{diam^2}} = \pi$$





$$\frac{\text{Area}}{\text{diam^2}} < \pi$$





$$\frac{\text{Area}}{\text{diam^2}} > \pi$$

Comments

The ideas of tangent spaces, affine connections and Riemannian metric allow us to do geometry (use geometric reasoning) in many problems.

e.g. Calculus of variations.

It is possible to geometry under a wide variety of metrics (distance) The metric can often be tailored to the problem

e.g. unbiased snakes

More general geometries are also possible by suitable definitions of affine connection.

e.g. affine differential geometry

You can get additional mileage by adding more structure to this

e.g. Lie groups