Continuity in Plain English

- Hemant D. Tagare

References:

[1] A First Course in Real Analysis - M. H. Protter
C. B. Morrey
Springer-Verlag Undergraduate Texts in Mathematics

[2] Real Analysis and Probability - R. B. Ash

IEEE Press

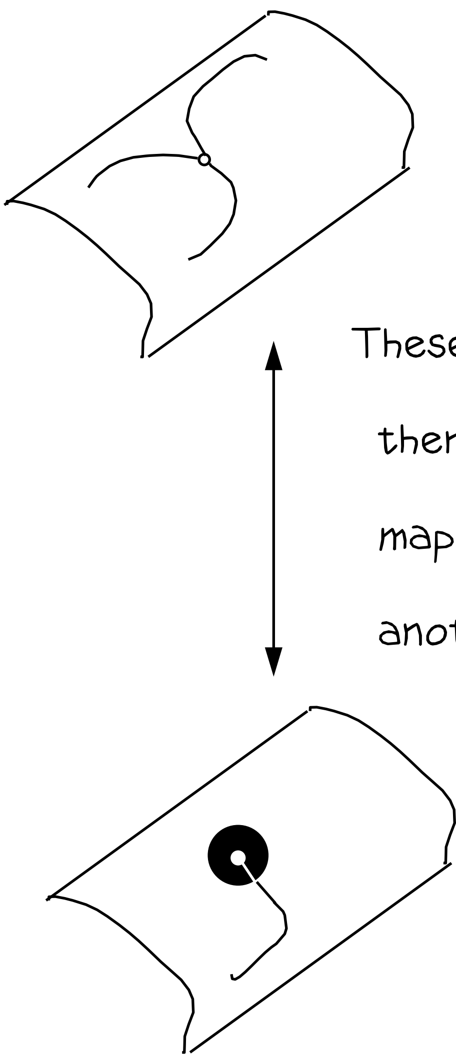
[3] Yet Another Introduction to Analysis - V. Bryant

Cambridge University Press

Why is continuity so important?

> We are interested in the following question:
>
> When are two spaces topologically equivalent ?

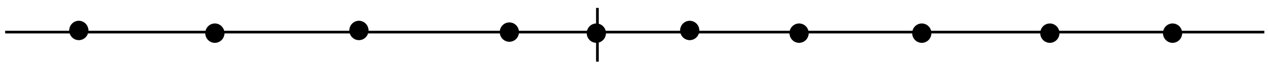
Continuity tells us how the space is connected.



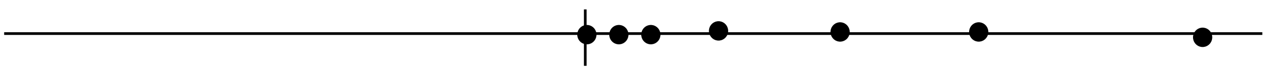
These spaces are different because there is no one-to-one map which will map convergent paths of one space to another

We don't want to assume path connectivity

We want to compare path connected as well as "discrete" spaces.



A = { 0.5 + n } U {0}



B = { 2^n } U {0}

There is a sequence in {2^n} which converges to 0, but there is no sequence in {0.5 + n} which converges to 0.

The spaces are different !

It is sufficient to consider convergence of sequences

=> Continuity of functions and open sets are of critical importance

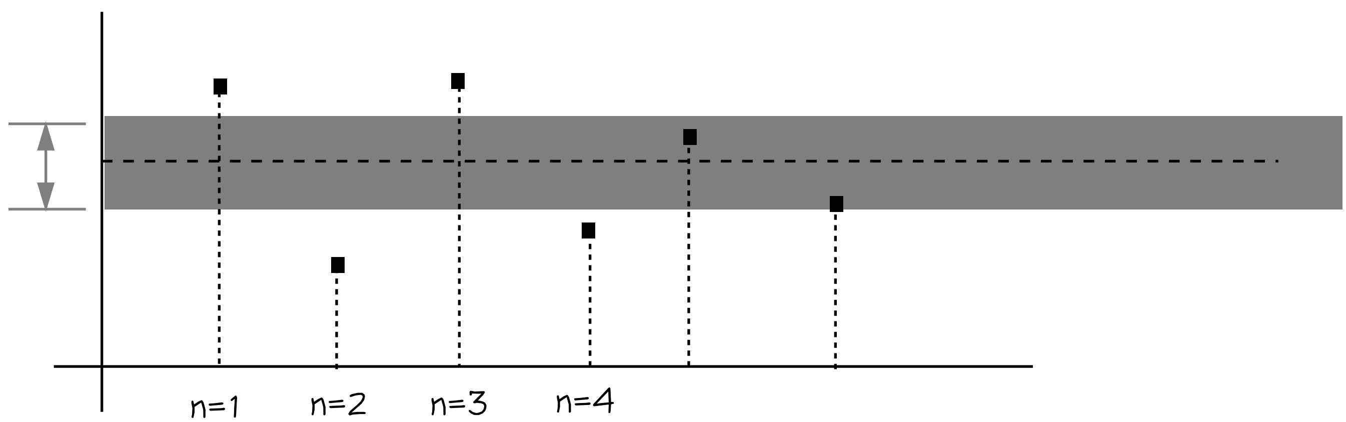
Convergence of a sequence of real numbers

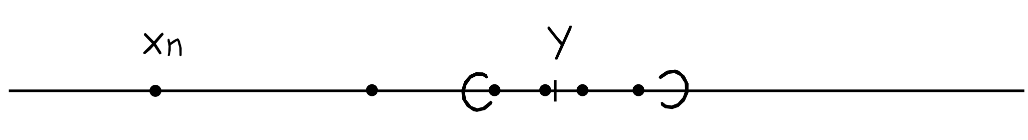
a.k.a Limit

We are given a sequence of real numbers

x_n , $n=1,2,\ldots\ldots$

we would like to know when they "tend" to a limit y





Limit of an infinite sequence

Intuition:

A number x is equal to y if $|x - y| = 0$

A number x is equal to y if $|x-y|$ is less than every positive real number.

every <--> any

A sequence x_n has limit y if $|x_n - y|$ is smaller than any positive real number for $n >$ (some) N.

Definition: A sequence x_n of real numbers has a limit y, if for every $\epsilon > 0$, there is an N such that $|x_n - y| < \epsilon$ for $n > N$.

$x_n \rightarrow y$

<u>Open Sets</u>

<u>Definition:</u> An <u>open interval</u> is the set of points

$\{x | a < x < b, a<b\}$

<u>Definition:</u> A point x c X is an interior point of the set if there is an open interval $I \subset X$ such that

$x \in I$.

<u>Definition</u>: A set X is open is every element of X is an interior point. The real line and the null set are open by definition.

<u>Theorem 0:</u> Except for the null set, every open set is a union of (possibly infinite number of) open intervals.

Open Sets

Theorem 1: [1] The union of open sets is open

[2] The intersection if finite number of open sets is open

Open sets and convergence of sequences

Definition: The trailing sub-sequence of a sequence x_n, $n >= 1$, is the sub-sequence x_n, where $n >= N >= 1$.

Theorem 2: A sequence converges to a point x if and only if every open set containing x contains some trailing sub-sequence.

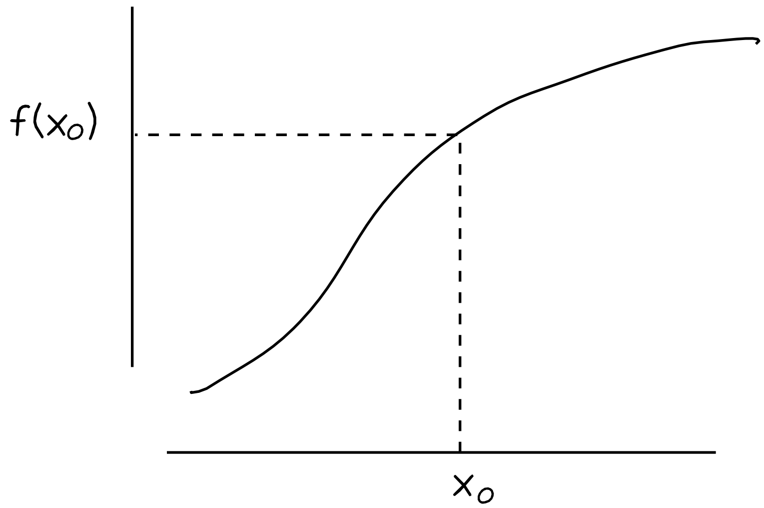
Theorem 3: A set X contains a trailing sub-sequence of every sequence convergent to $x \in X$ if and only if it contains an open set containing x.

Proof: The "if" part is trivial.

Suppose that every open interval (set) containing x has a point w which is disjoint from X. Consider the sequence of open intervals $O_n = \{u, |u-x| < 2\text{^}(-n)\}$, $n >= 1$, and let w_n be the point in O_n which is not in X.

Then, the sequence w_n is disjoint from X, but tends to x. Contradiction.

Continuity

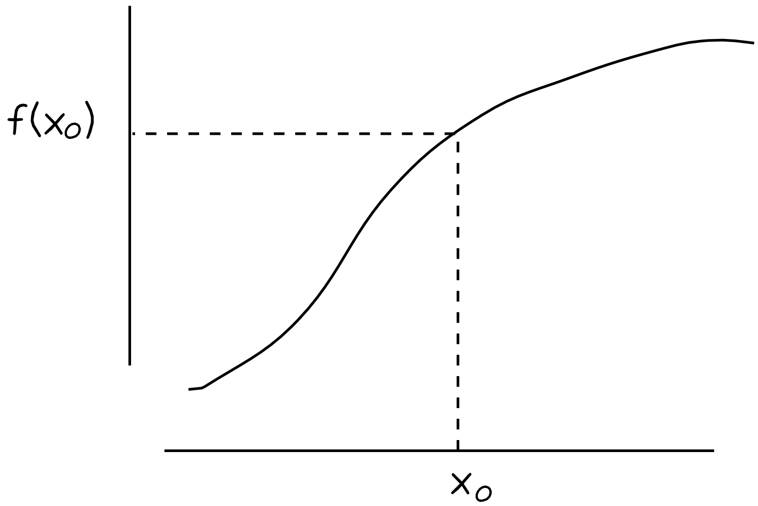


Definition: A function $f: R \to R$ is continuous at x_o if

$f(x_n) \to f(x_o)$ for every $x_n \to x_o$.

Theorem 4: Let O be an open set containing $f(x_o)$. Then, f is continuous at x_o if and only if $f^{-1}(O)$ contains an open set containing x_o.

Continuity



Definition: A function f:R->R is simply called continuous if it is continuous everywhere.

Theorem 5: A function is continuous if and only if the inverse image of every open set is open.

Proof: The "if" part -> choose an open set in the range and apply theorem 4 to every point in the set.

The "only if" part -> Let 0 be the open interval of length 2^(-m) centered at $f(x_o)$. Then $f^{-1}(0)$ is open and contains x_o. Let x_n be any sequence that converges to x_o. The sequence has a trailing edge in $f^{-1}(0)$. Thus, the sequence $f(x_n)$ has a trailing edge in 0. Since this is true for every m>=1 , it follows that $f(x_n) \to f(x_o)$. Thus f is continuous at x_o. But x_o is arbitrary,

Topology

Note that the theorem does not guarantee that the (forward) image of an open set is open.

What does this mean?

The theorem guarantees that image of every convergent sequence is convergent

It does not guarantee that every convergent sequence in the range space is the image of some convergent sequence in the domain space.

Thus, there may be more ways of getting to $f(x_0)$ in the range space than there are ways to getting to x_0 in the domain space.

=> The two spaces may be topologically different.

Topology

How can we set up a relation between two spaces so that every convergent sequence in one space is the image of a convergent sequence in the other?

Do it with a function f:

[1] f must be one-to-one. => f and f^{-1} must exist

[2] Every point of both spaces must have a corresponding point

=> f must be onto.

[3] The image of every open set (under f and f^{-1}) must be open

=> f and f^{-1} must be continuous.

This is a homeomorphism.

But we have define open sets and continuity.

Topology

In the typical style of mathematics if and only if theorems of simple objects become definitions of the abstractions of the objects.

OPEN SET:

It is sufficient to use theorem 1 plus a part of the definition.

CONTINUITY:

Sufficient to use theorem 5 as a definition.

HOMEOMORPHISM: (HOMEOMORPHIC SPACES)

A homeomorphism simply defines what we mean by the intuitive sentence "the structure of two spaces is similar".

It does not (necessarily) tell us what that structure is.

For this, we need topological invariants.