Further Manifolds in Plain English

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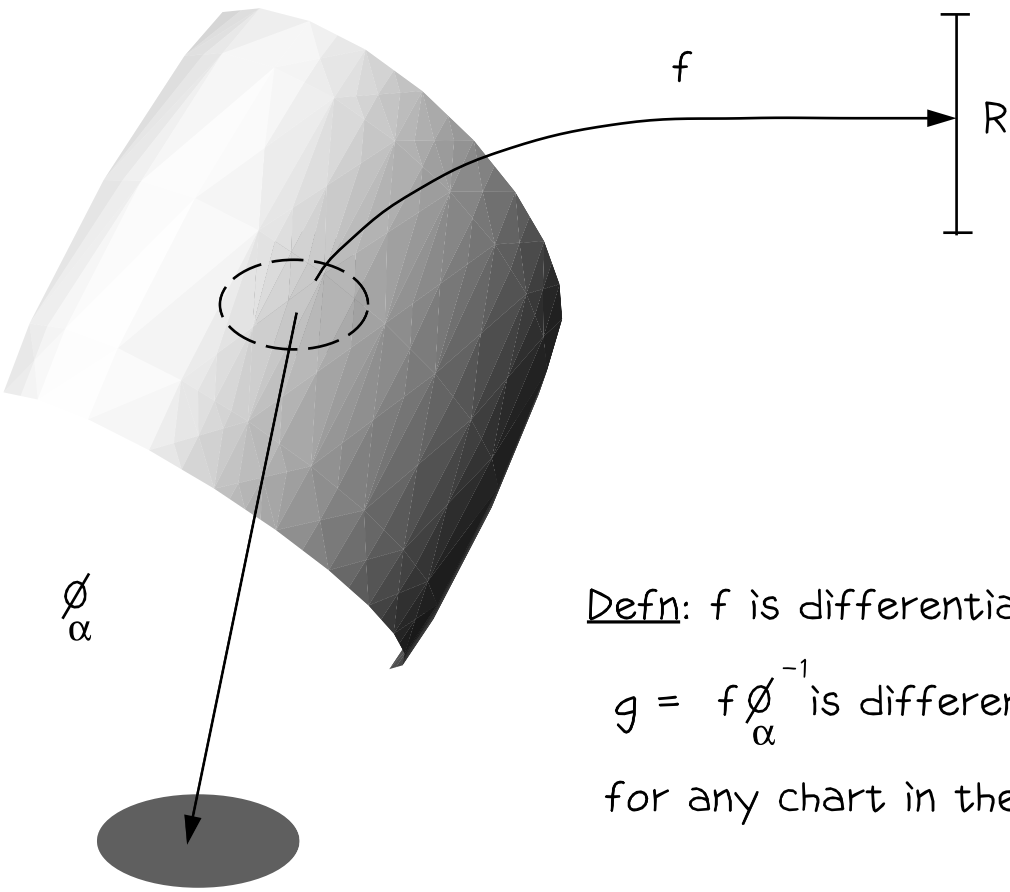
Differential Structure of (on) a Manifold

Just as open sets define the sense in which a convergent sequence can be constructed in a topological space,

the differential structure defines the sense in which derivatives can be taken on a "vector-like" space.

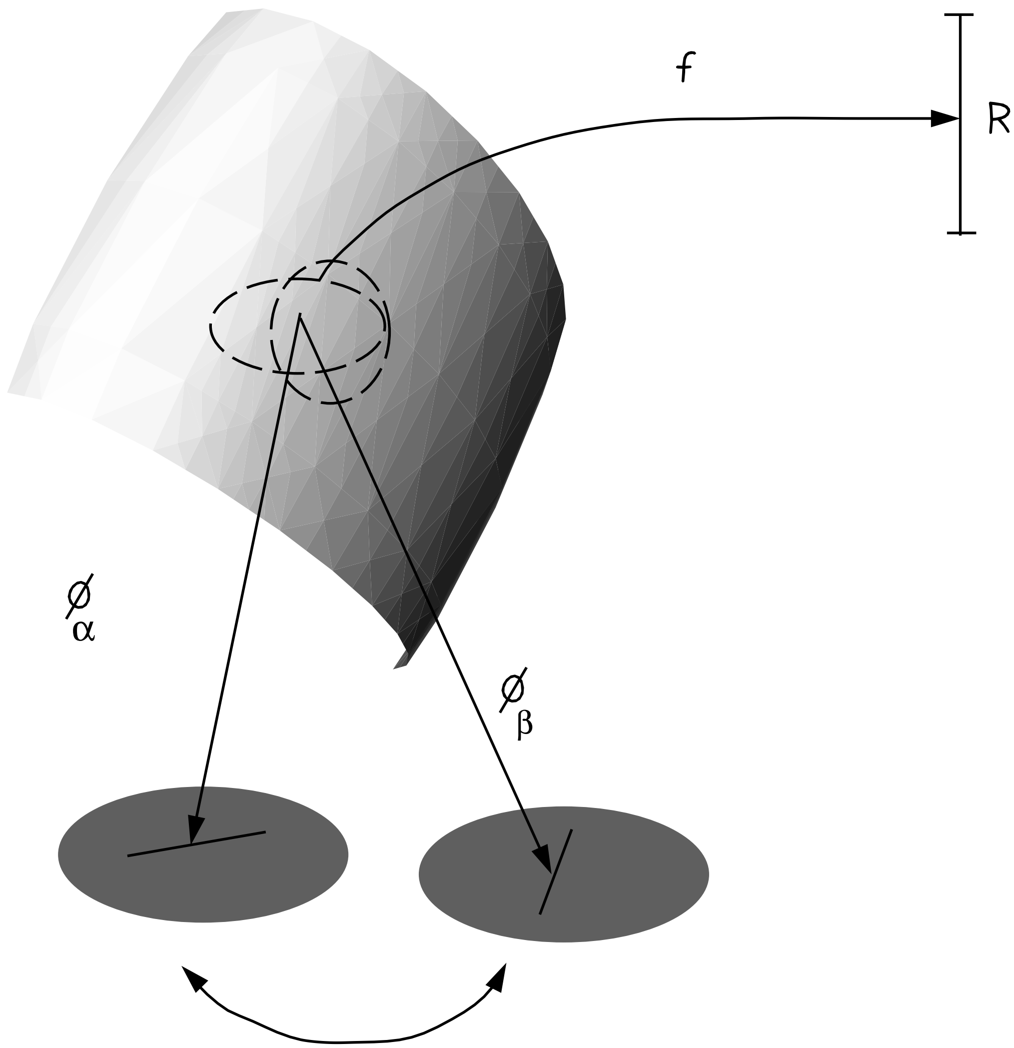


Defn: f is differentiable if

$g = f\,\phi_\alpha^{-1}$ is differentiable

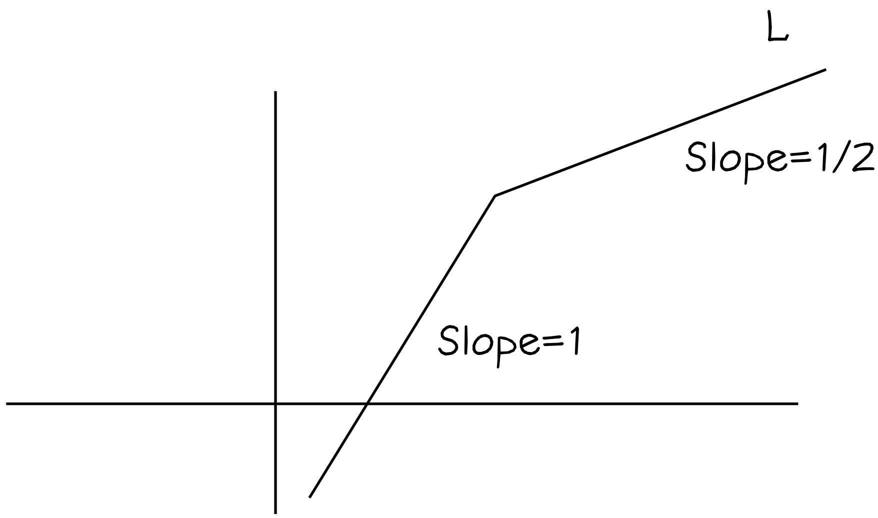
for any chart in the altas

Differential Structure of (on) a Manifold



The derivative along corresponding lines exists and is given by the change of variables rule (chain rule).

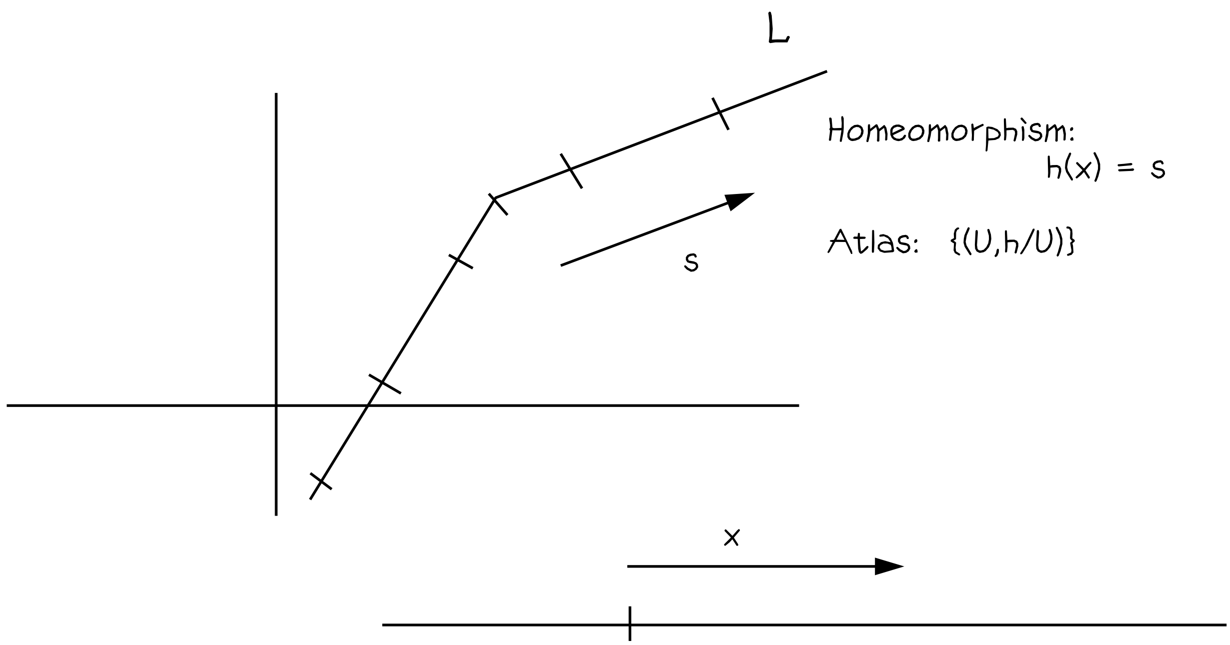
Differential Structure



Atlas 1: Use the Euclidean length of L to diffeomorphically map R onto L

In doing this we have ignored the co-ordinate axis

Differential Structure 1

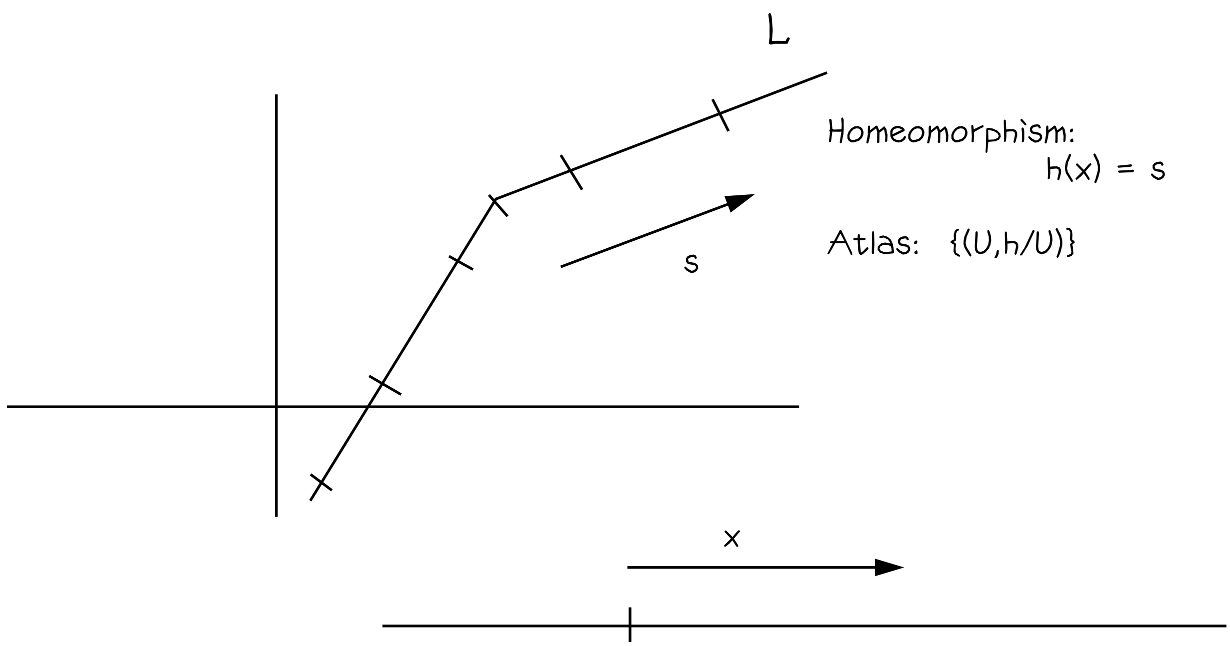


Atlas 1: Use the Euclidean length of L to diffeomorphically map R onto L

This is differentiable manifold

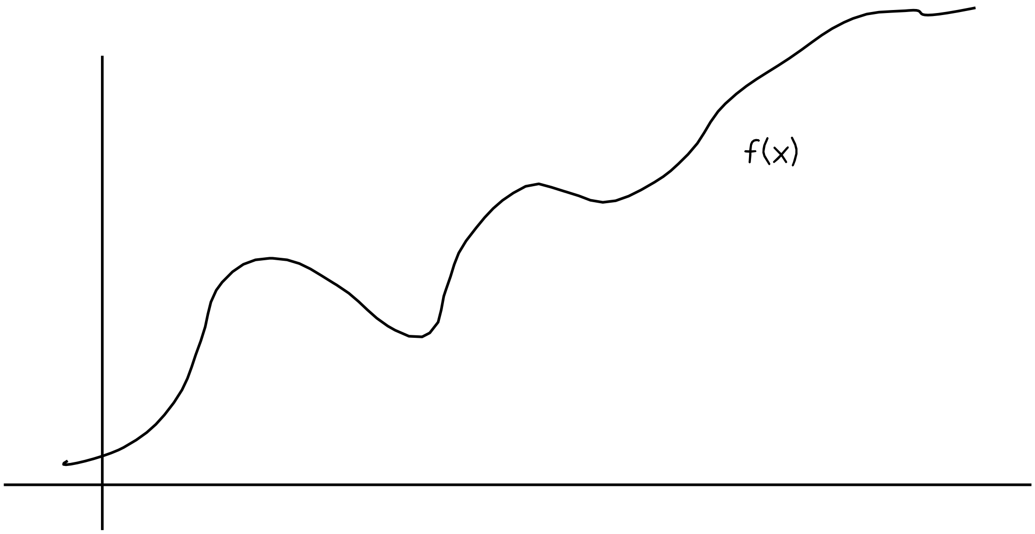
Note: The definition of a manifold treats the manifold by itself, and NOT as it may be embedded in a bigger space

Differential Structure 1



Any real differentiable function f(x) gives rise to a differentiable function on L according to the recipe f(s)

Graph of a function from R -> R



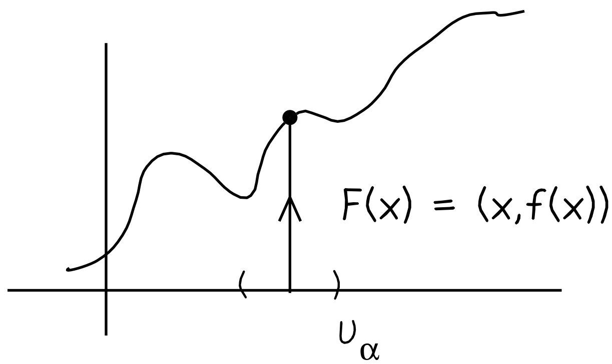
Defn: The graph of a function f:R -> R is the set

$$G_f = \{ (x,f(x)) \mid x \subset R \}$$

Theorem: G_f is a 1-manifold in R X R that is homeomorphic to R

Graph of a function

Theorem: G_f is a 1-manifold in R X R that is homeomorphic to R



Proof: First show homeomorphism, then differentiable manifold.

homeomorphism: We will show that F:x->(x,f(x)) is a homeomorphism

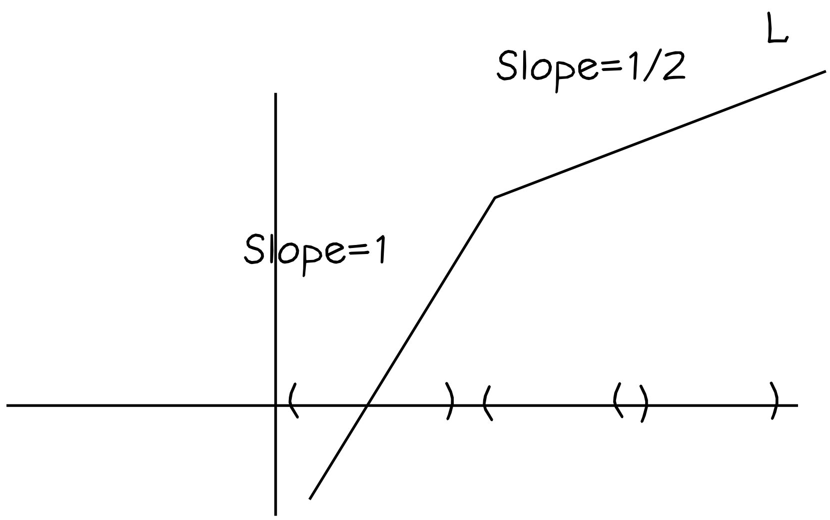
F is one-to-one and onto. (why?)
F is continuous because x->x and x->f(x) are continuous. (why ?)
F^{-1} is just Π_1, which is continuous by subspace topology.

=>F is a homeomorphism.

differentiable manifold: Atlas $\{(U_\alpha, \emptyset_\alpha)\}$ where U_α is open in R and $\emptyset_\alpha = F/U_\alpha$.
In $U_\alpha \cap U_\beta$ $\emptyset_\alpha^{-1} \emptyset_\beta = I$ which is a diffeomorphism. QED.

f is only required to be continuous

Differential Structure 2



Atlas 2: All open sets of the x-axis, with the function F.

Note: All functions on L which are differentiable with respect to Atlas 1 ARE NOT differentiable with respect to this atlas.

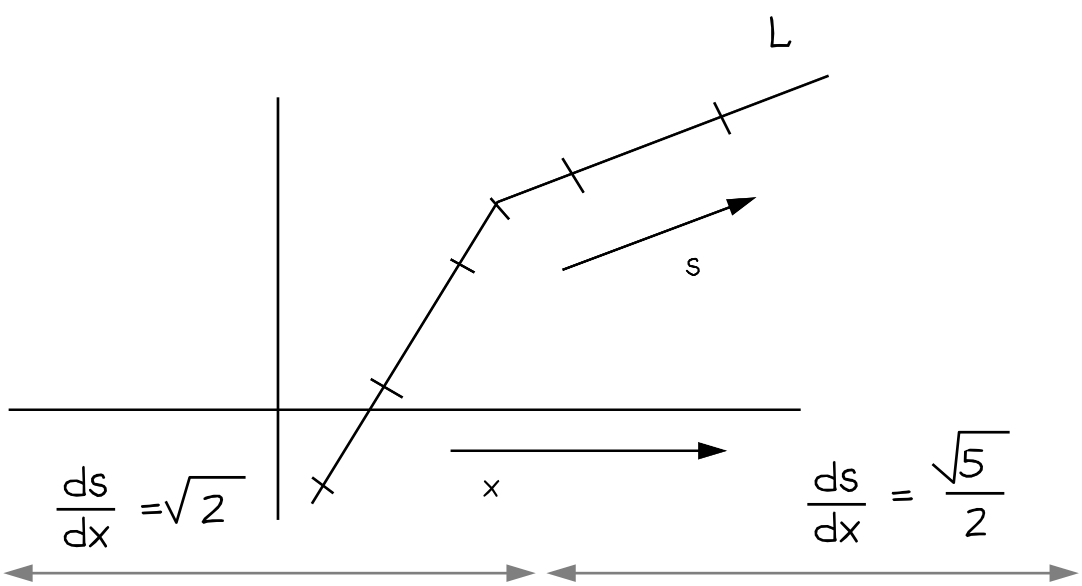
These are two different differentiable structures on the same topological space L

Differential Structure 2

Characterize functions which are differentiable w.r.t. Atlas 2

in terms of Atlas 1

Clearly, these will not be differentiable w.r.t. Atlas 1.



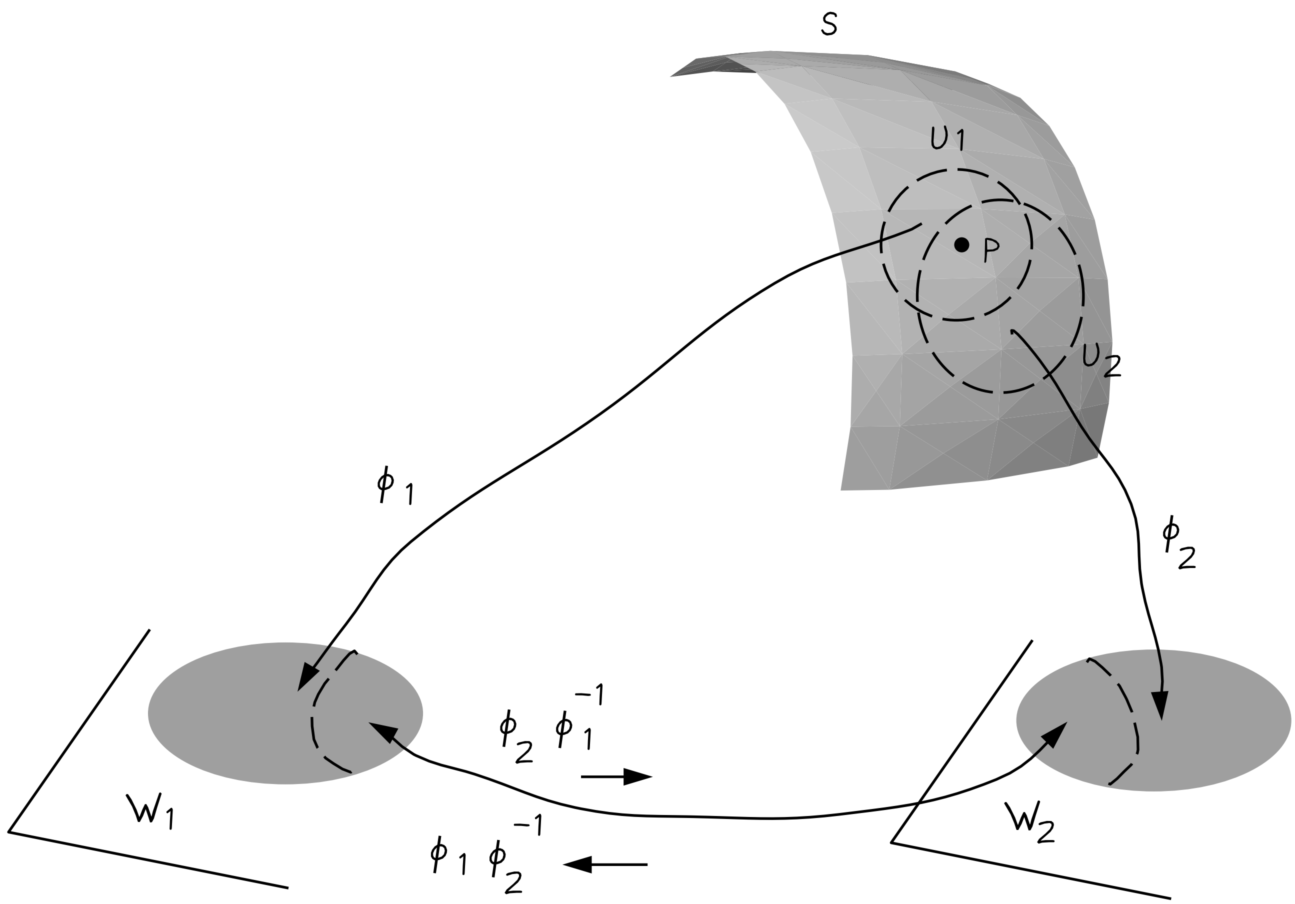
$$\frac{ds}{dx} = \sqrt{2} \qquad \frac{ds}{dx} = \frac{\sqrt{5}}{2}$$

$$\frac{df}{dx} = \frac{df}{ds}\frac{ds}{dx}$$

All continuous functions f(s)
which are differentiable except at 0
and,

$$\frac{\lim_{s \to 0^-} f'(s)}{\sqrt{2}} = \frac{\lim_{s \to 0^+} f'(s)}{\frac{\sqrt{5}}{2}}$$

Differential Structure Manifold

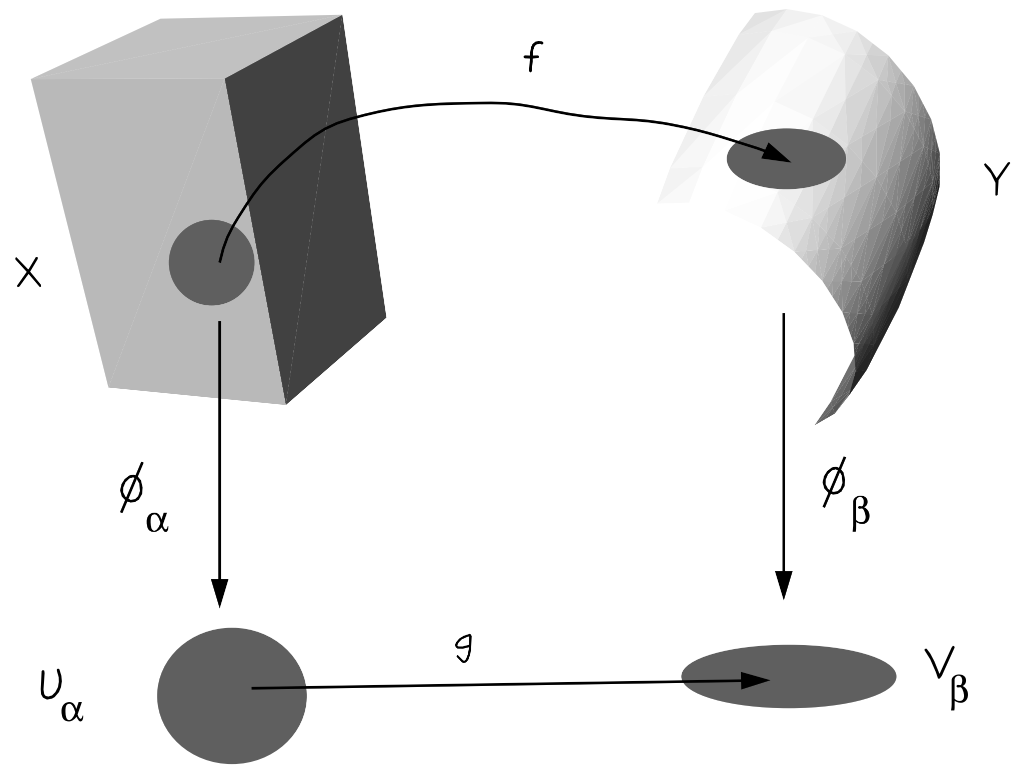


The differential structure is NOT incidental to a manifold it is infact the sense in which a derivative is defined on the manifold.

It is as important to a manifold as open sets are to a topological space.

The same set with different differential structures are different manifolds.

Functions between manifolds



Note: X and Y need not have the same dimension

Defn: A function f:X->Y from manifold X to manifold Y associates a single element of Y with each element of X.

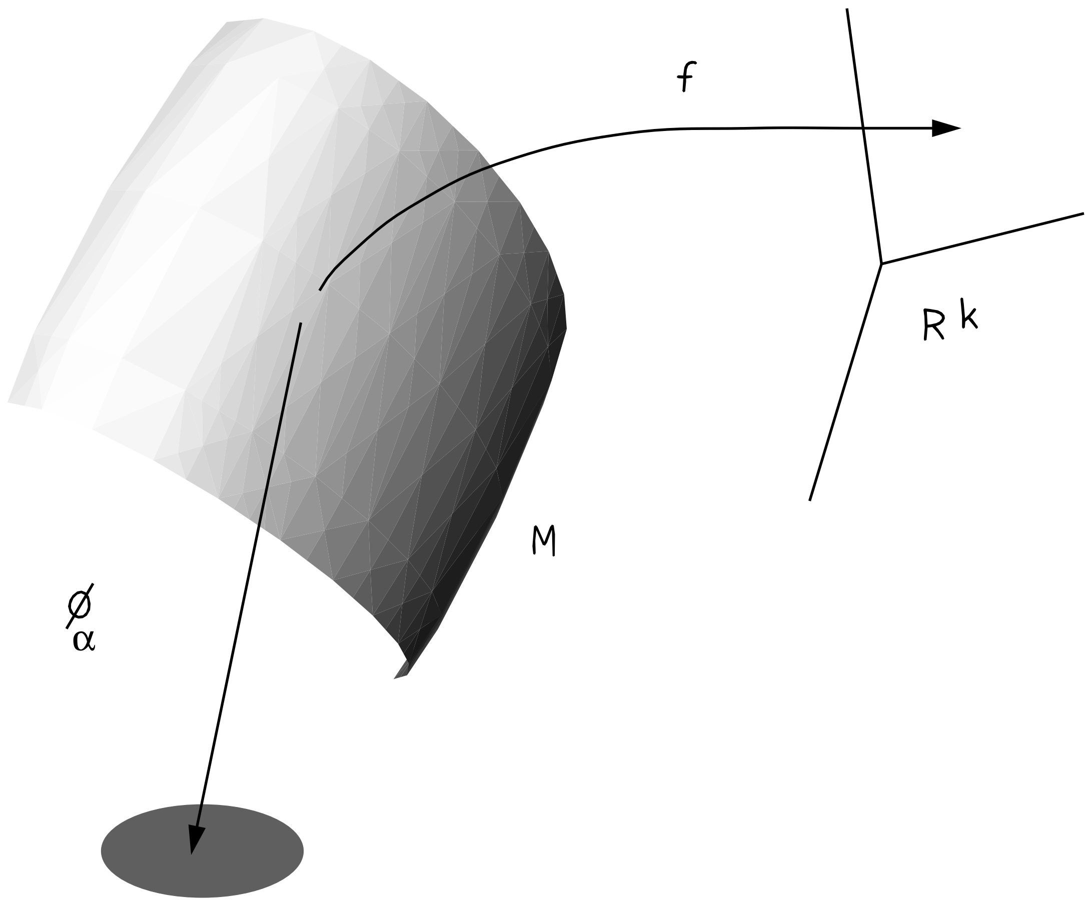
Defn: The function f:X->Y is continuous if it is continuous is the topological sense

Defn: A function f is differentiable if its associated function

$$g = \phi_\beta \, f \, \phi_\alpha^{-1}$$

is differentiable in the standard sense for all charts in the atlas

Regular Embedding of a Manifold in R^k



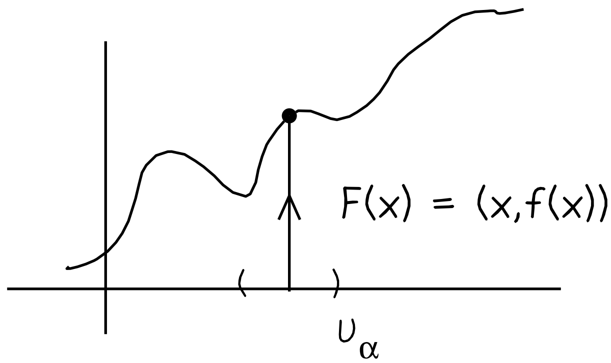
Defn: A manifold M is said to be regularly embedded in R

if there is an injective function f:M->R such that

the derivative $d(f\ \phi_\alpha^{-1})$ of the compostive function has

full rank in the entire atlas.

Graph of a function



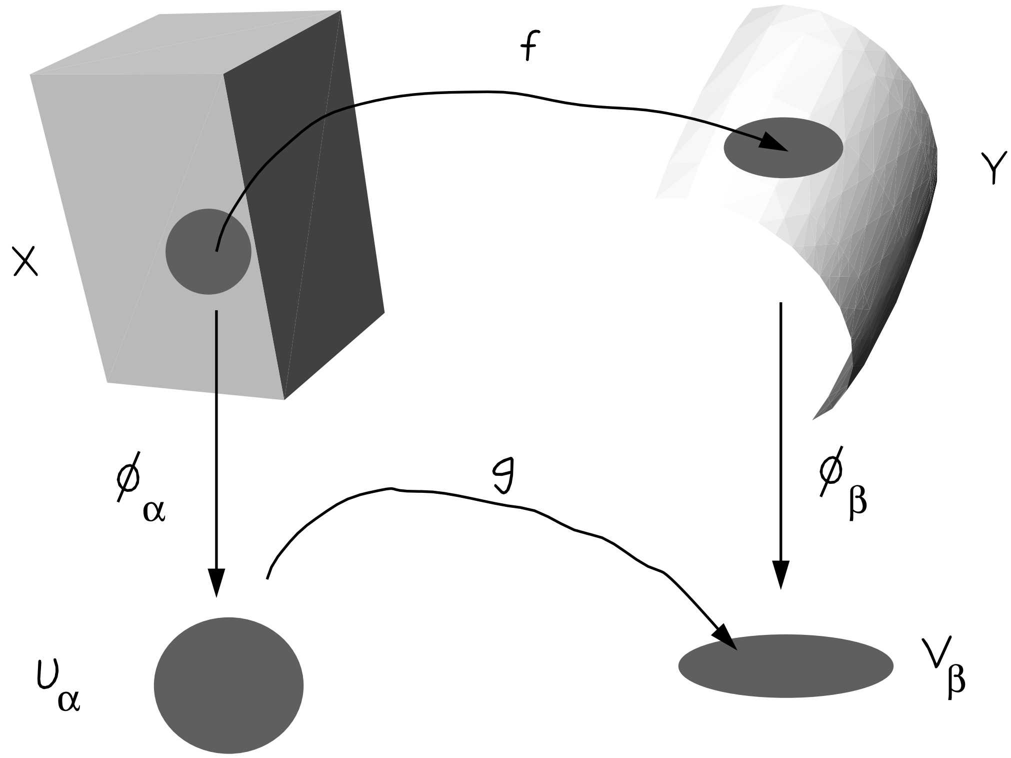
Theorem: G_f is a 1-manifold in R X R that is homeomorphic to R

Further, it is a regular curve as a subset of R X R if and only if f is differentiable.

Proof: The embedding function from G_f is $g((x,f(x)) = (x,f(x))$.

Therefore $Fg(x) = (x,f(x))$ and its derivative exists (and has full rank) if and only if f is differentiable.

Functions between manifolds



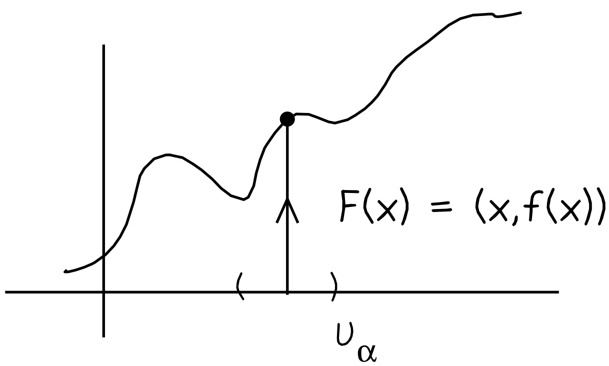
Defn: The graph of a function f:X -> Y is the set

$$G_f = \{ (x,f(x)) \mid x \subset X \}$$

What does the graph of a continuous function look like?

Functions between manifolds

Theorem: G_f is a 1-manifold in R X R that is homeomorphic to R



Proof: First show homeomorphism, then differentiable manifold.

homeomorphism: We will show that F:x->(x,f(x)) is a homeomorphism

F is one-to-one and onto. (why?)
F is continuous because x->x and x->f(x) are continuous. (why ?)
F^{-1} is just Π_1 , which is continuous by subspace topology.

=>F is a homeomorphism.

differentiable manifold: Atlas $\{(U_\alpha,\emptyset_\alpha)\}$ where U_α is open in R and $\emptyset_\alpha = F/U_\alpha$.
In $U_\alpha \cap U_\beta$ $\emptyset_\alpha^{-1}\emptyset_\beta = I$ which is a diffeomorphism. QED.

Notice that this proof does not use special properties of R !!!

Functions between manifolds

> **Theorem:** G_f is a n-manifold in X X Y that is homeomorphic to X

Proof: First show homeomorphism, then differentiable manifold.

homeomorphism: We will show that F:x->(x,f(x)) is a homeomorphism

F is one-to-one and onto. (why?)
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> This holds for ANY pair of manifolds
>
> X and Y

Functions between manifolds

Special Case:

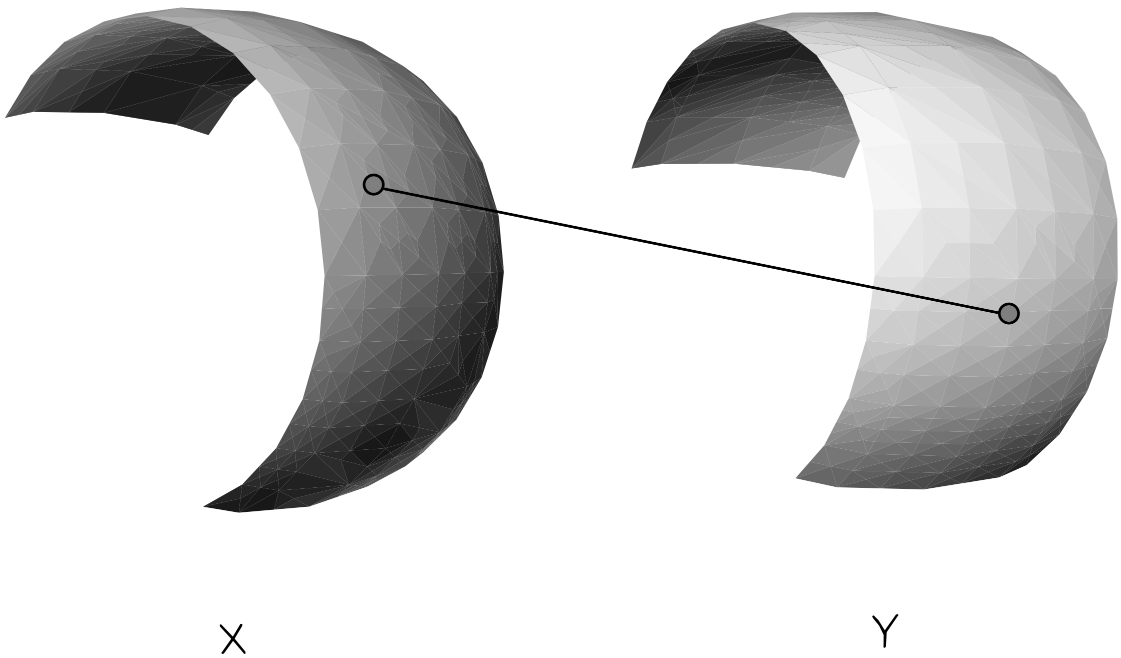
f is a homeomorphism: X and Y are homeomorphic manifolds.

Theorem: The graph of a homeomorphism is a differentiable manifold which is homeomorphic to the base manifolds.

f is a diffeomorphism: X and Y are diffeomorphic manifolds

Theorem: In addition to the above the graph is regular.

Correspondences between Manifolds



Defn: A correspondence between X and Y is the Φ in X X Y such that $\Pi_1(\Phi) = X$ and $\Pi_2(\Phi) = Y$.

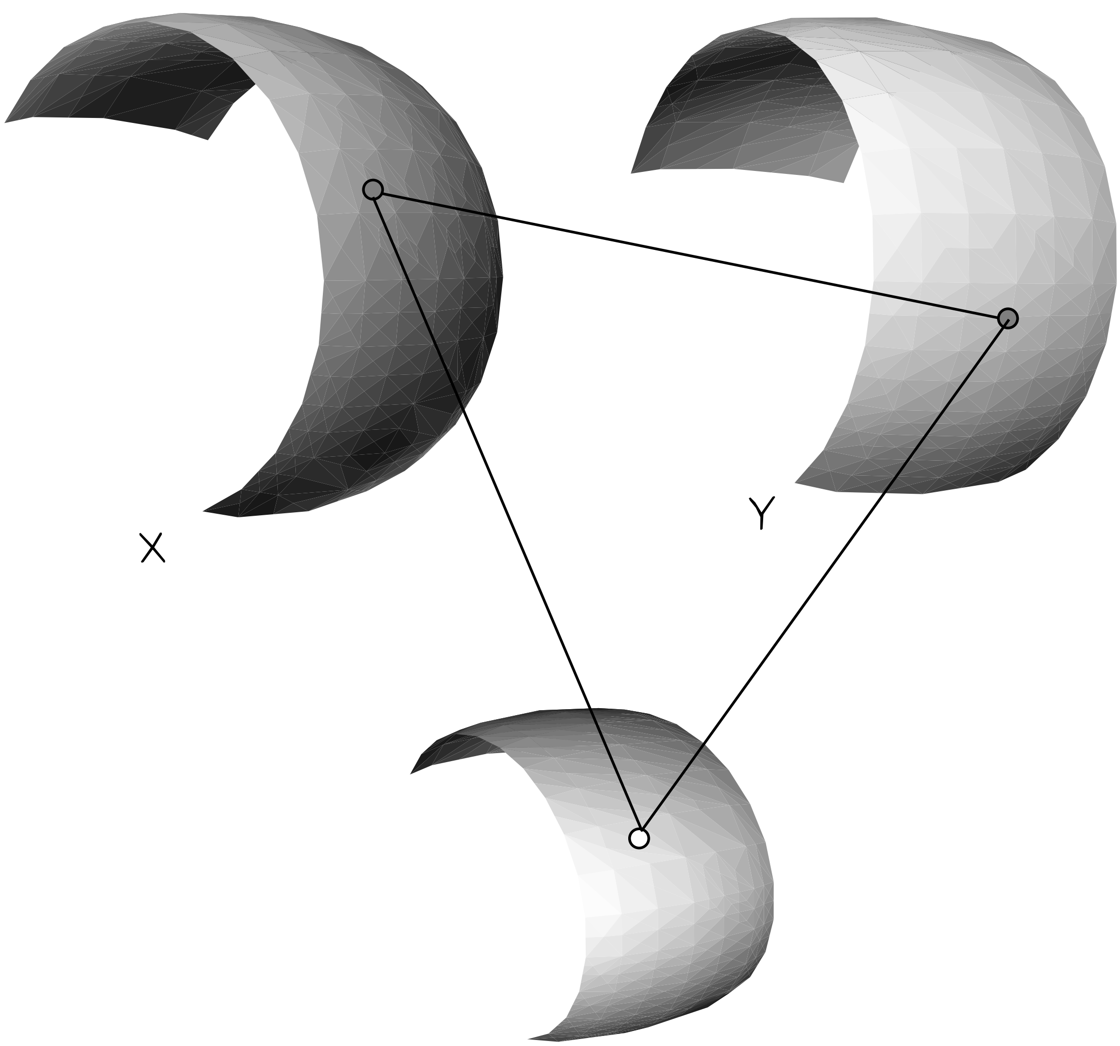
Defn: A function f:X->Y gives the correspondence {(x,f(x))}.

IT IS JUST THE GRAPH OF THE FUNCTION !

Correspondences between Manifolds

<u>Theorem:</u> Any homeomorphic correspondence between any two manifolds is a differentiable manifold which is homeomorphic to the base manifolds

If in addition, the corresppondence is diffeomorphic then (as a manifold) is regular.



Φ as a subset of X X Y

Conclusions

Try to think in terms of open sets, topology, manifolds and differentiable structures.

You can get very powerful results !!