Topology in Plain English

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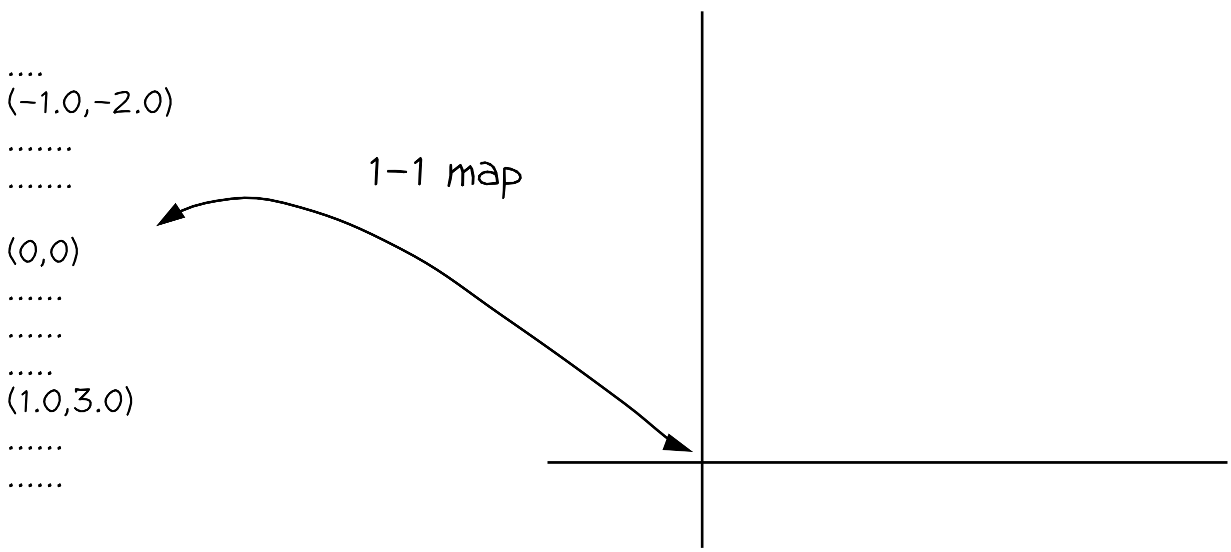
Why topology?

Need it for the definition of a manifold

Since topological spaces are more general than manifolds, we can appreciate what differential geometry can and cannot do.

This is a very informal introduction

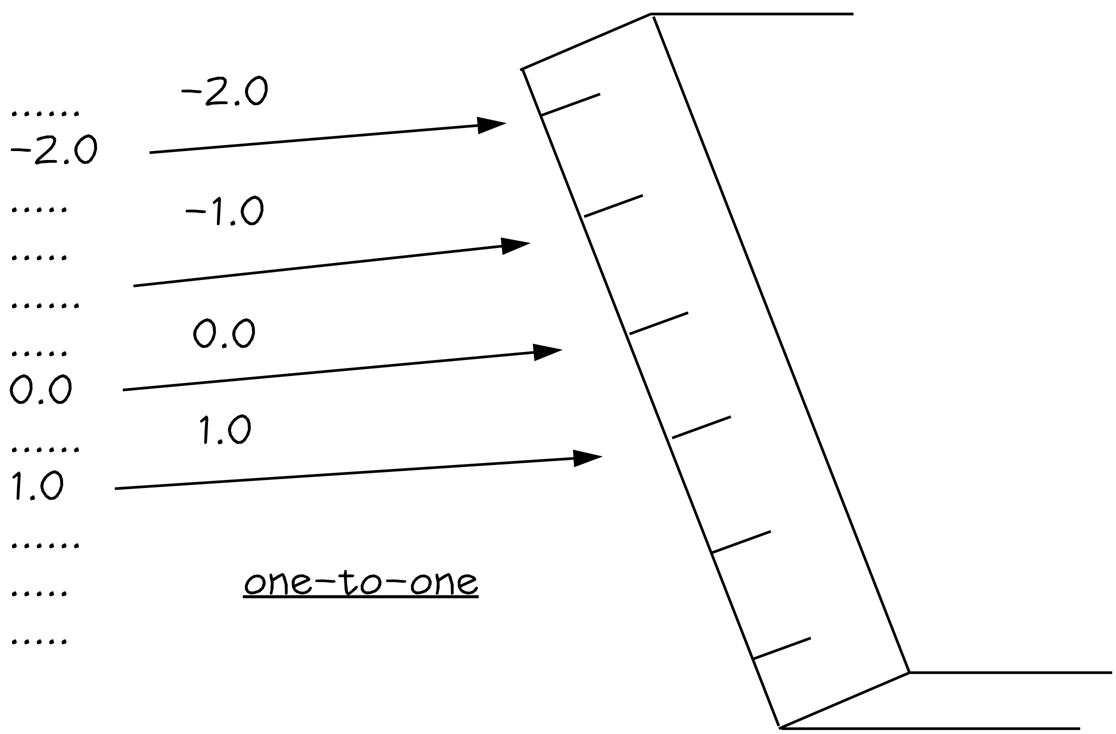
Topology as generalization of Descarte's insight



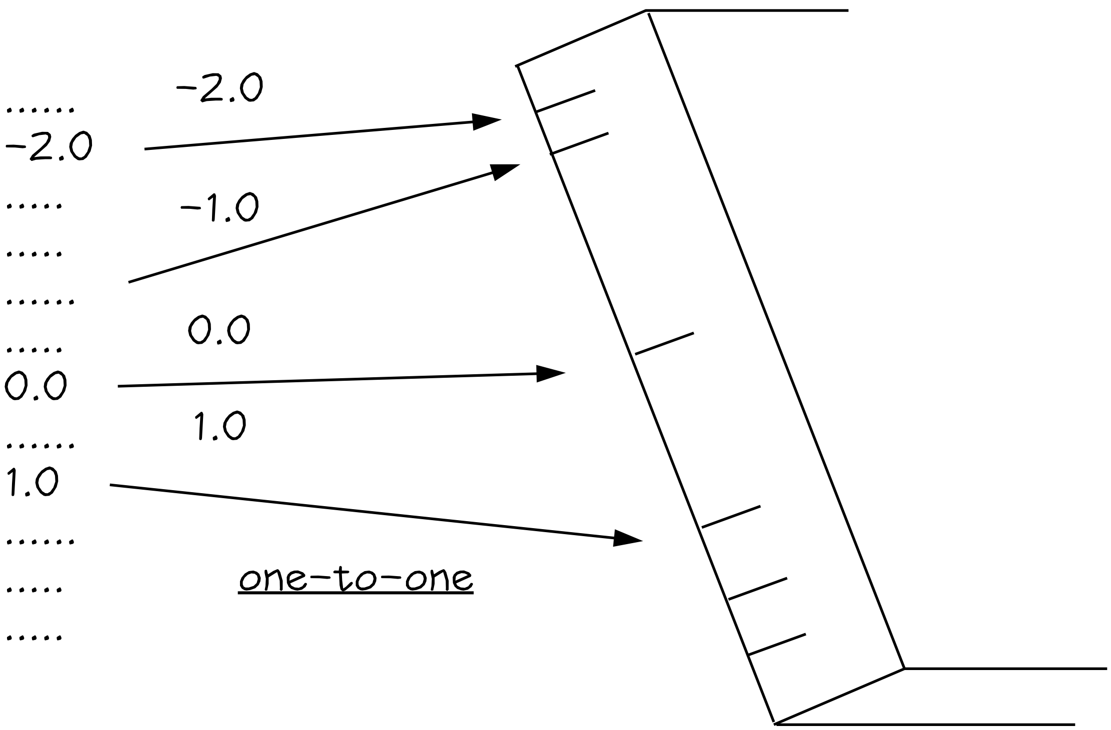
The set of pairs of real-numbers ↔ Points in a plane

These are the "same" thing

Topology as generalization of Descarte's insight

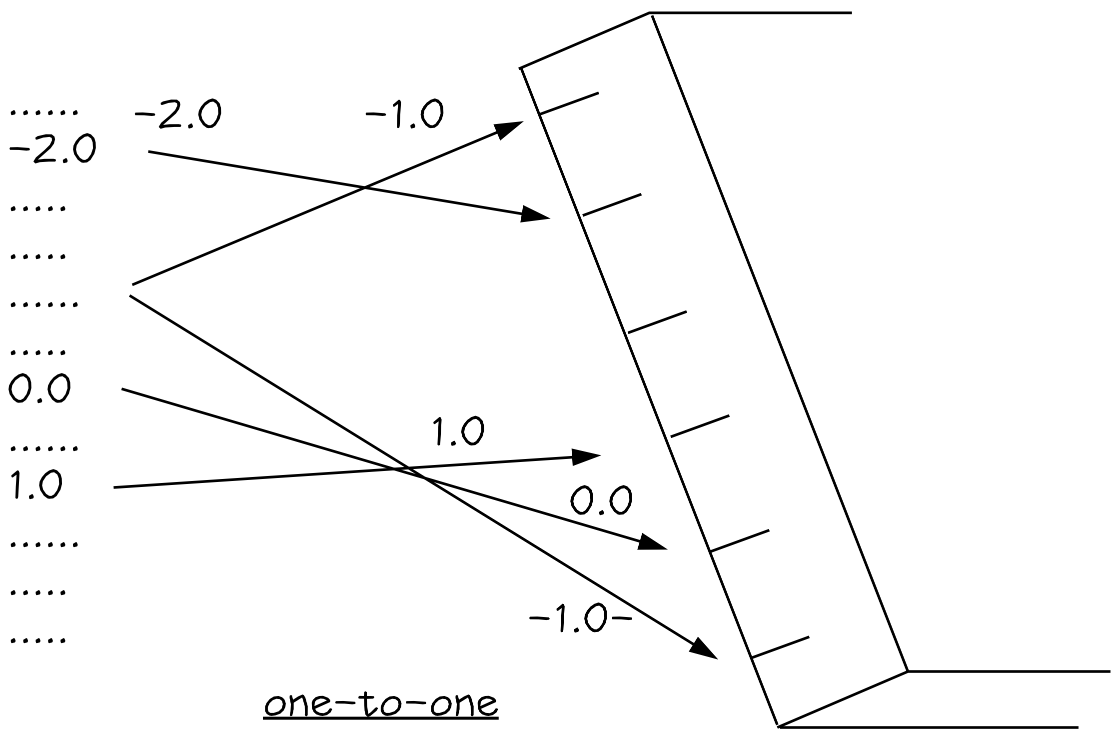


Topology as generalization of Descarte's insight



Changes distance (metric) but not much else

Topology as generalization of Descarte's insight



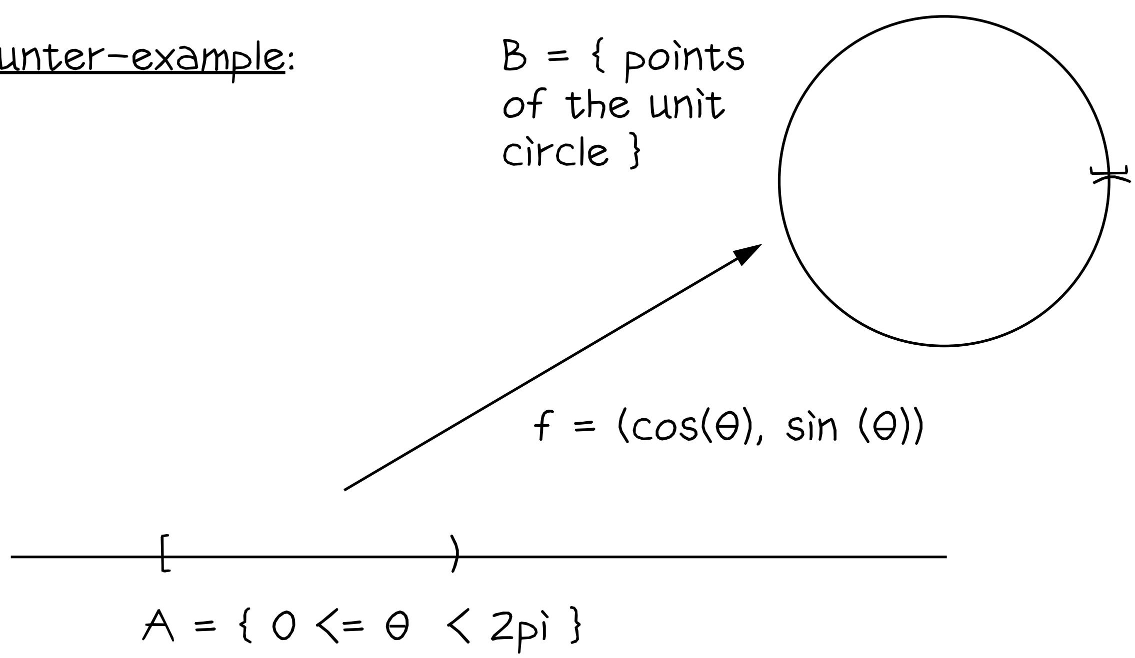
Absolutely key point:

This mapping is not useful because it is not continuous

Topology as generalization of Descarte's insight

Attempt 1: Two sets A and B are topologically equivalent if there is a ``continuous'' one-to-one and onto function f:A -> B.

Counter-example:



Topology as generalization of Descarte's insight

Definition: Sets A and B are topologically equivalent (or homeomorphic) if there is a one-to-one and onto function $f: A \rightarrow B$, such that f and f^{-1} are continuous.

f is called a homeomorphism

Continuity is absolutely crucial in topological issues

Continuity for ordinary functions

Defn: A function f:R -> R is continuous at x_0 if for any

$e > 0$ there is an $d > 0$ such that

$| f(x_0) - f(x) | < e$ for all $|x_0 - x| < d$.

Defn: A set N is a neighborhood of x if N contains an open

interval containing x.

Defn: A function f:R -> R is continuous at x_0 if the inverse

image of any neighborhood of $f(x_0)$ contains a

neighborhood of x_0

Topological Space

Hausdorf, 1914

Defn: A topological space is a set X, and for eachpoint x of X a nonempty collection of subsets of X (called neighborhoods of x) which satisfy

(a) x lies in each of its neighborhoods,

(b) The intersections of two neighborhoods of x is also a neigborhood of x,

(c) If N is a neighborhood of x and U is a subset of X containing N, then U is a neighborhood of x,

(d) If N is a neighborhood of x, and $\mathring{N}$ is the set $\{z \in N \mid N \text{ is a neighborhood of } z\}$, then N is a neighborhood of x.
($\mathring{N}$ is called the interior of N)

The same set X with different neighborhood systems is a different topological space.

Alternate Definitions are sometimes easier

From Open sets:

Defn: A non-empty collection of subsets of X is a collection of open sets if

(a) Any (not finite) union of open sets is open,

(b) Any finite intersection of open sets is open,

(c) X and the empty set are open.

Defn: N is a neighborhood of a point x is N contains an open set which contains x

Giving a set a topology means choosing a system of neighborhoods or open (or closed) sets

Continuity

Definition: A function f from a topological space A to a topological space B is continuous at $x \in A$ if the inverse image of any open set containing $f(x)$ is open.

Topology of A

Topology of B

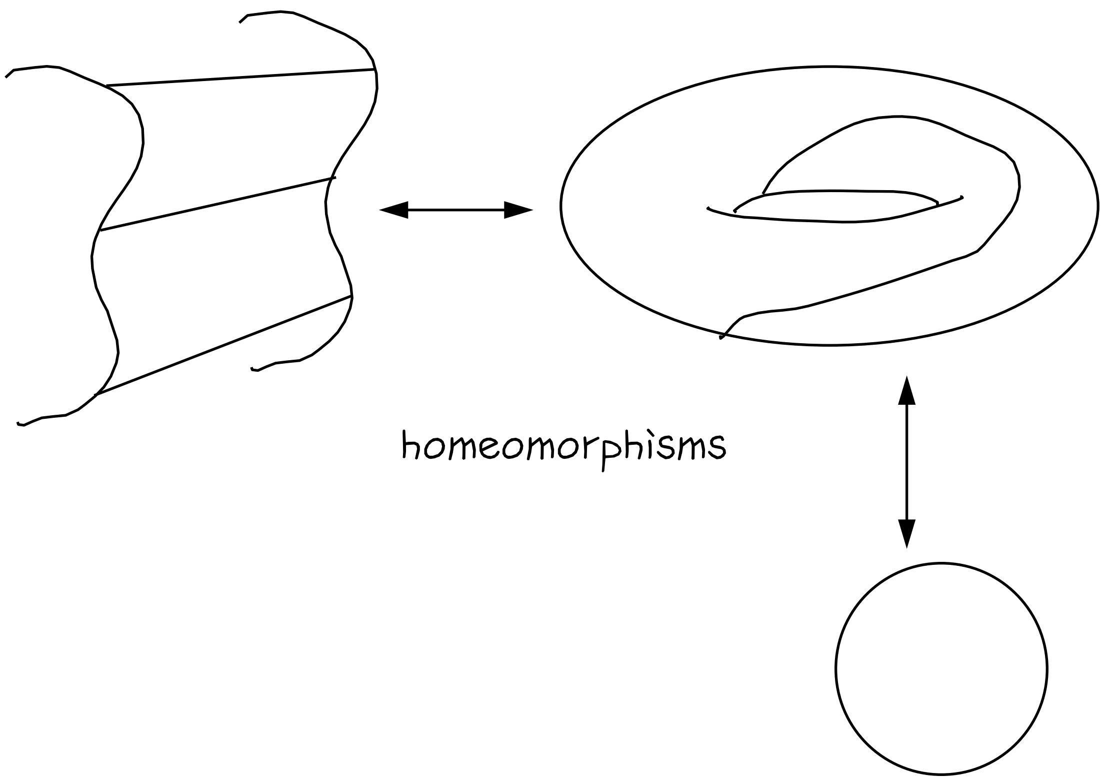
Note: A function may not be continuous if you change the topology.

Definition: Two topological spaces are homeomorphic if there there is an onto and one-to-one function from one space to the other, such that the function and its inverse are continuous

Why should we care about this?

The parametrization of any new mathematical object is really a homeomorphism from some familiar structure to the new object (More later)

This is really how we ``understand'' new mathematical objects



Finding a (useful) topological space which is not homeomorphic to a known topological is a major mathematical event

Some examples

Let R be the set of real numbers

An open interval $O = \{x | a < x < b, a \neq b\}$

An open set is any union of open intervals

This is called the usual topology on R

Let R be the set of real numbers

Any subset of R is an open set

Every point x has a single neighborhood $\{x\}$

This is called the discrete topology on R

Every function is continuous in the discrete topology

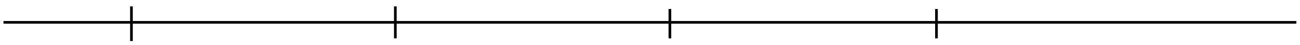
Let R be the set of real numbers

A subset O is open if $R - O$ contains a finite number of elements, or is equal to R

This is called the finite-complement topology.

It is really strange

Some examples



$A_u = \{x | \ x = u + n, \ \ o<= u < 1, \ n \text{ integer}\}$

$A = \{A_u\}$

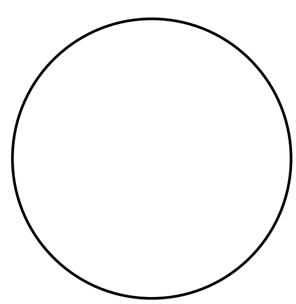
with open sets defined as the the union of subsets

$$B_{r,s} = \bigcup_{r < u < s} A_u$$

What is this topology?

Just a circle

with unit circumference



Key Point

Topological equivalences are very hard to grasp intuitively.

We need formal techniques for doing this:

[a] We need to indentify useful ways in which topological spaces appear (are created) in applications

Product spaces, Identification Spaces, Covering Spaces etc.

[b] We need ways of calculating topological invariants of such spaces

(It is often easier to determine when topological spaces are not equivalent, then when they are equivalent)

Homotopy and homology groups of spaces.

Product Spaces

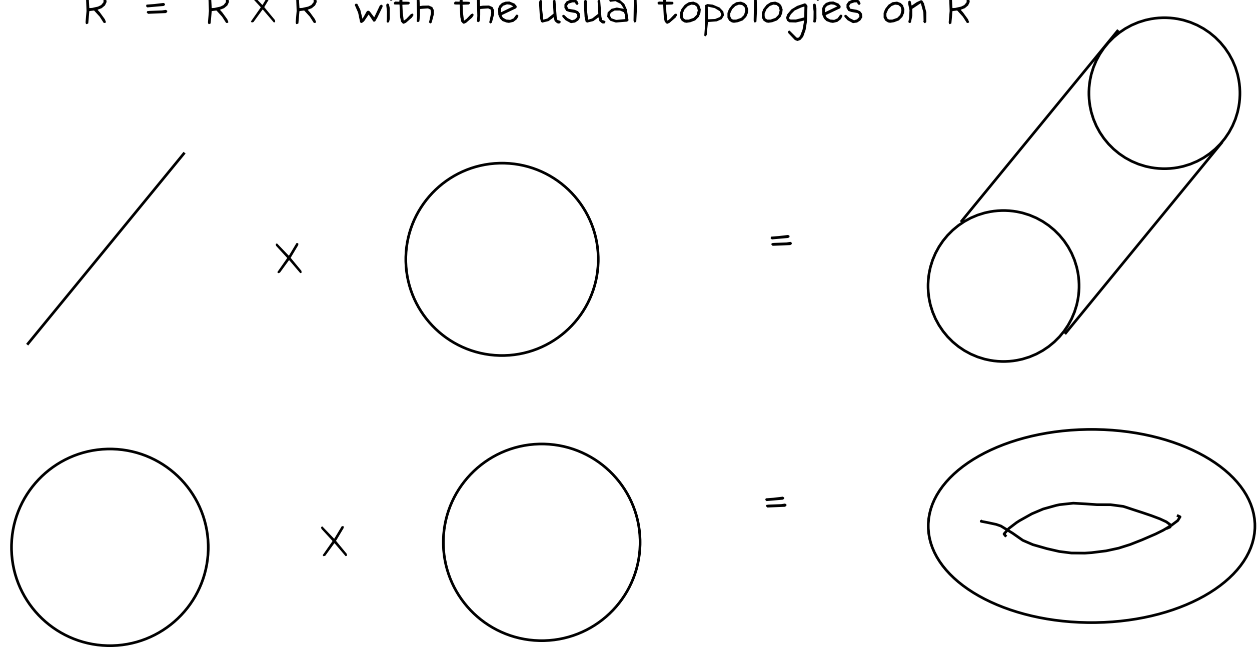
Defn: The product space A X B of two topological spaces is the set of all ordered pairs

A X B = { (a,b) | a c A b c B },

with the following system of open sets:

A subset W of A X B is open if it can be written as a union of sets of the type U X W, where U is open in A and W is open B.

R^2 = R X R with the usual topologies on R



Descarte's brilliant idea

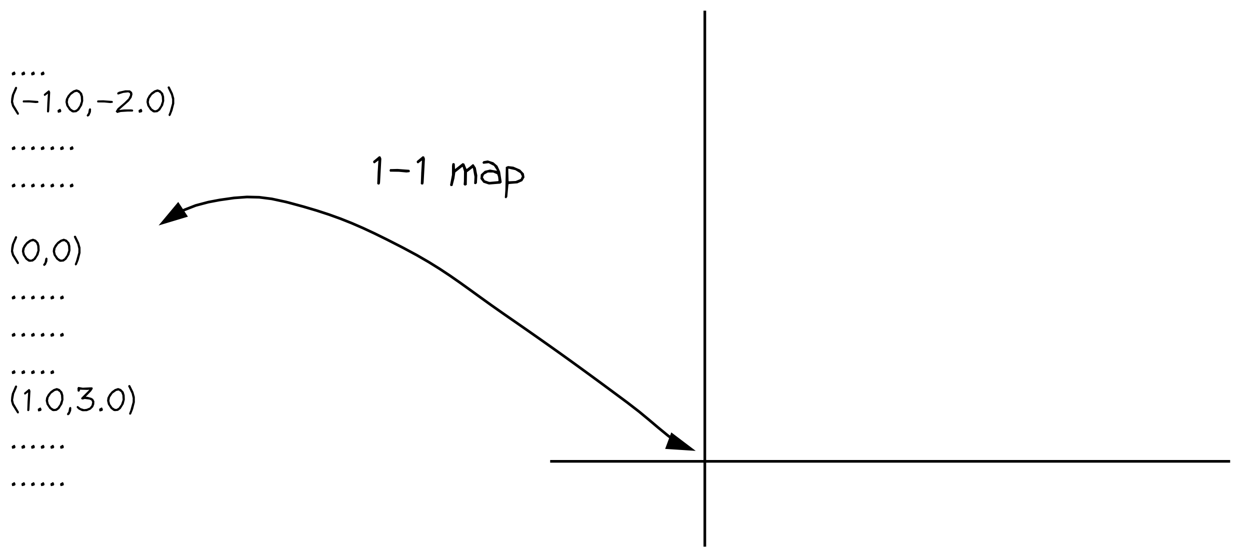
Take R^2 with its usual topology

Take the plane with its usual topology

(Open sets are unions of open discs)

Impose a co-ordinate system on the plane

This givs a one-to-one and onto function f from R^2 to the plane

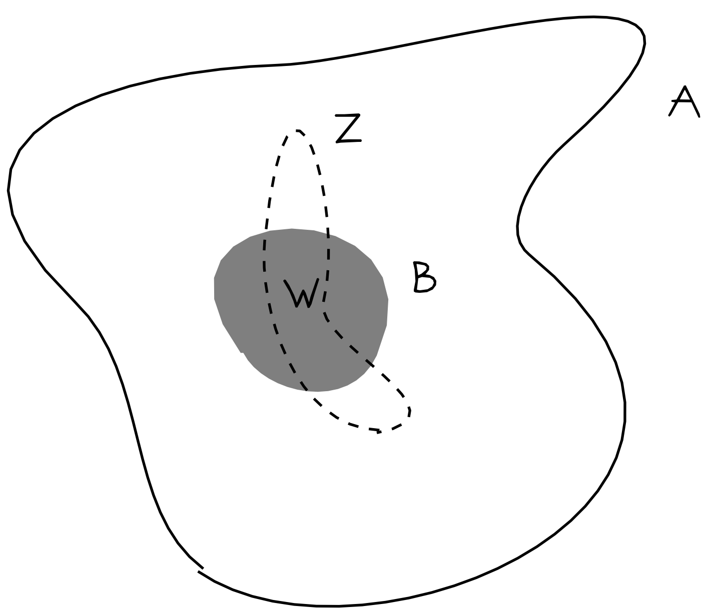


f is not just one-to-one and onto but is actually a homeomorphism

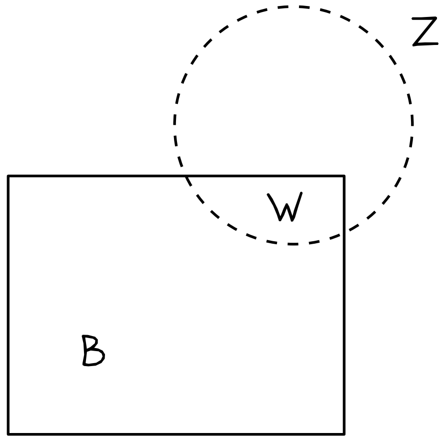
Subspace Topology

Defn: Let B be a subset of a topological space A. The subspace topology on B is the topology that W is an open set in B if and only if $W = B \cap Z$, where Z is open in A.

Warning: W may not be open in the topology of A



Subspace Topology

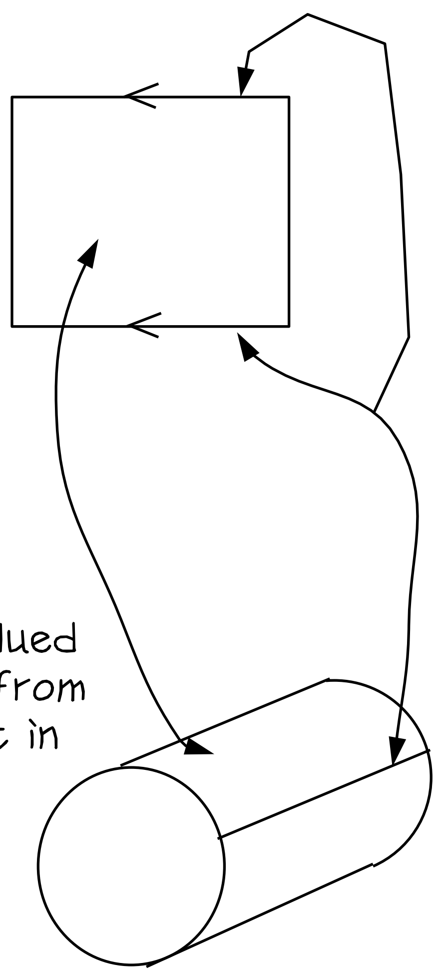


B = closed unit square in the plane

W is not open in the plane but is open in B

Identification Spaces

Intuition: If you glue opposite sides of a square
you get a cylinder



Every point on the glued edge comes from a pair of points in the square

Every point not on the glued edge comes from a single point in the square

Every point in the square goes to exactly one point in the cylinder

Identification Spaces

Defn: Let X be a topological space

Let P be a family of disjoint subsets of X such that

$\cup P = X$,

Let Y be a set whose points are members of P

Let p: X -> Y be the map that takes every point of X

to the subset containing Y

Let a subset O of Y be open if and only if $p^{-1}(O)$ is

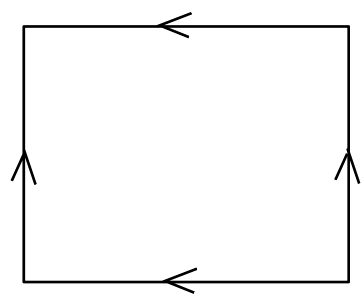
open in Y

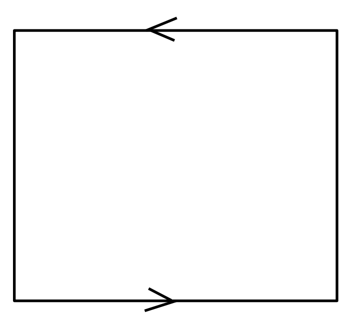
Under these conditions Y is a topological space with the

identification topology. (Y is an identification

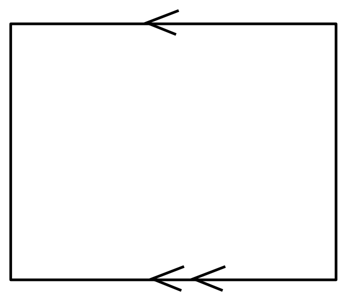
space)

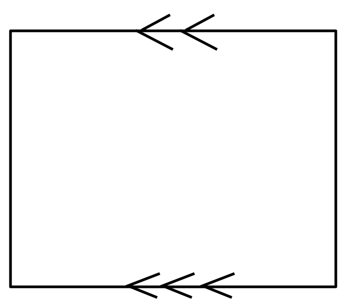
Identification Spaces

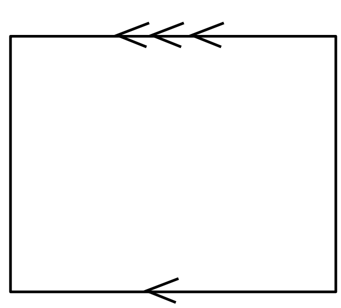




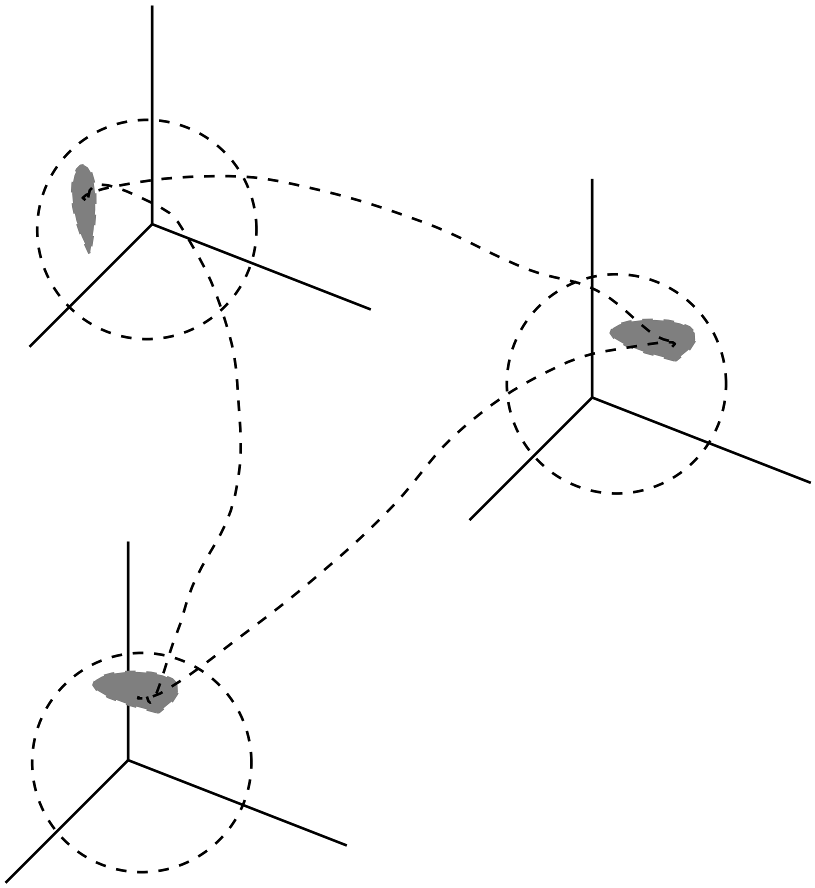
Identification Spaces







Identification Spaces



An extention of this idea gives us a

manifold

A really cool theorem

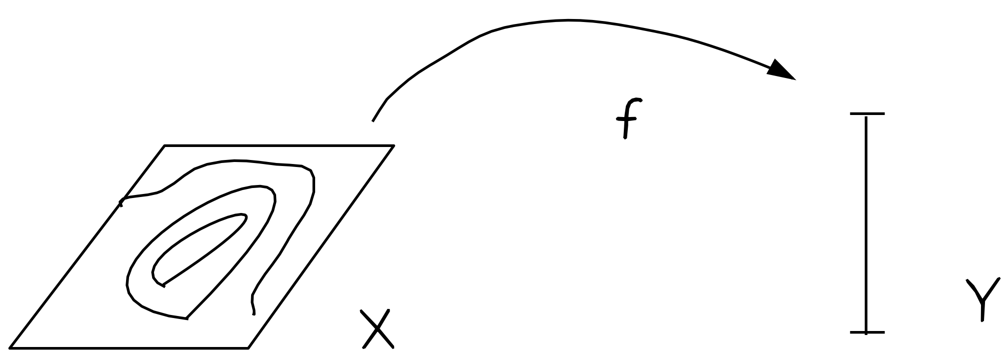
Let f: X -> Y be an onto and continuous function,

(and suppose that Y has the largest topology for which f is continuous), then f partitions X according to $f^{-1}(y)$, $y \in Y$.

The technical condition is satisfied if X is compact and Y is Hausfdorf

Let Y* be the identification space associated with the partition.

<u>Theorem</u>: Y* is homeomorphic to Y



Further Topology

Just as we generalized the notions of open sets and continuous functions, we can generlize the notions of connected and compact sets

Connectivity is a topological invariant

Invariants are important for showing when two topological spaces are not homeomorphic.

The key to doing all of this is to generalize common notions by using a set of formal properties.

surface
derivatives
vector

Knowing which properties to use takes genius.