Further Topology in Plain English

Hemant D. Tagare

Keep this in mind:

Specifying open sets is equivalent to specifying convergent sequences

Key Point

Topological equivalences are very hard to grasp intuitively.

We need formal techniques for doing this:

[a] We need to indentify useful ways in which topological spaces appear (are created) in applications

Product spaces, Identification Spaces, Covering Spaces etc.

[b] We need ways of calculating topological invariants of such spaces

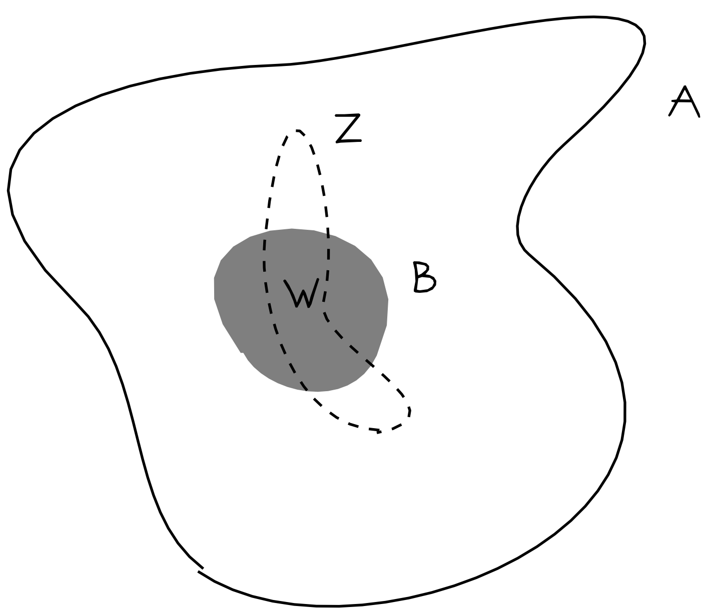
(It is often easier to determine when topological spaces are not equivalent, then when they are equivalent)

Homotopy and homology groups of spaces.

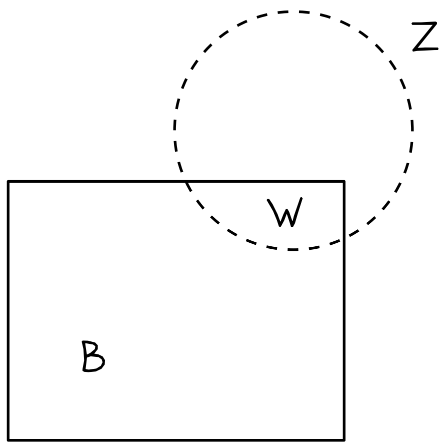
Subspace Topology

Defn: Let B be a subset of a topological space A. The subspace topology on B is the topology that W is an open set in B if and only if $W = B \cap Z$, where Z is open in A.

Warning: W may not be open in the topology of A



Subspace Topology



B = closed unit square in the plane

W is not open in the plane but is open in B

Product Spaces

Defn: If A and B are sets, then their product A X B is the set

$$A \times B = \{ (a,b) \mid a \in A, \ b \in B \}$$

$$\Pi_1 : A \times B \to A \qquad \Pi_1((a,b)) = a$$

$$\Pi_2 : A \times B \to B \qquad \Pi_2((a,b)) = b$$

What we want:

A sequence (a_n, b_n) is convergent in A X B if and only if a_n is convergent in A and b_n is convergent in B.

A set $O \subset A \times B$ is open if and only if its projections on A and B are open.

Product Spaces

Defn: The product space A X B of two topological spaces is the set of all ordered pairs

A X B = { (a,b) | a c A b c B },

with the following system of open sets:

A subset W of A X B is open if its projections on A and B are open.

Theorem: The projection functions are continuous

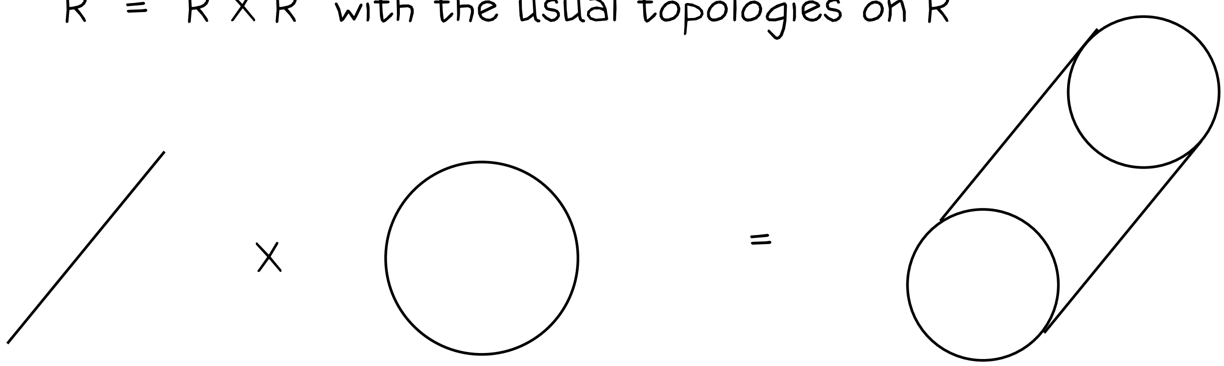
Product Spaces

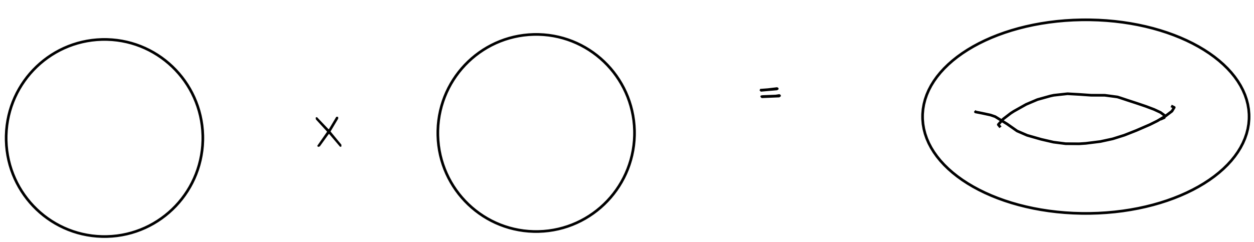
How should we visualize this?

For a fixed $a \in A$, the set $\{(a,b), b \in B\}$ looks just
like B

For a fixed $b \in B$, the set $\{(a,b), a \in A\}$ looks just like
like A

$R^2 = R \times R$ with the usual topologies on R





Descarte's brilliant idea

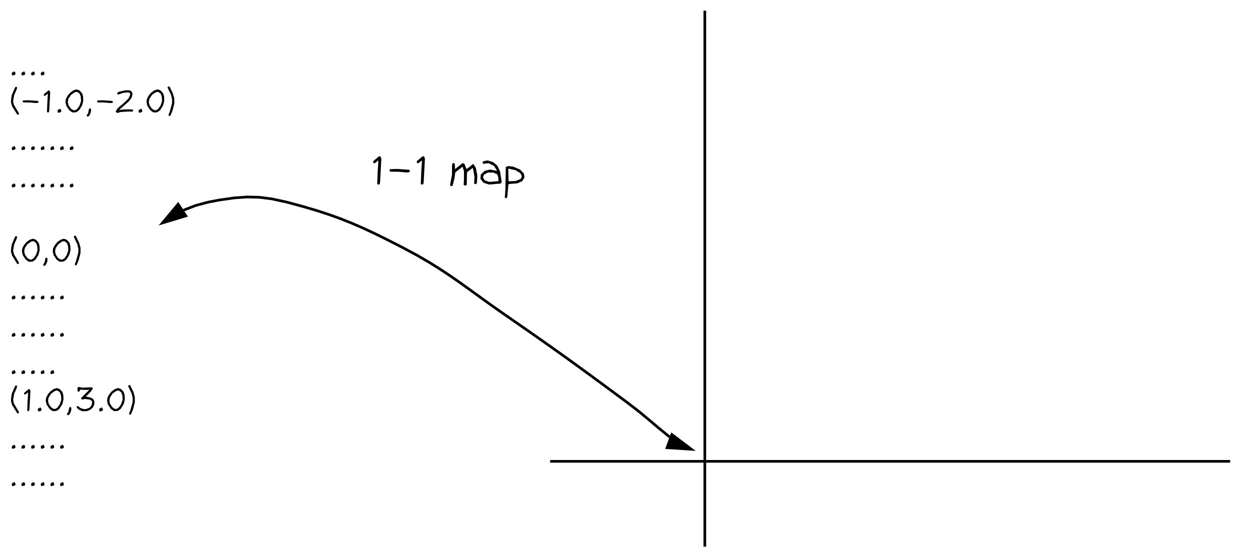
Take R^2 with its usual topology

Take the plane with its usual topology

(Open sets are unions of open discs)

Impose a co-ordinate system on the plane

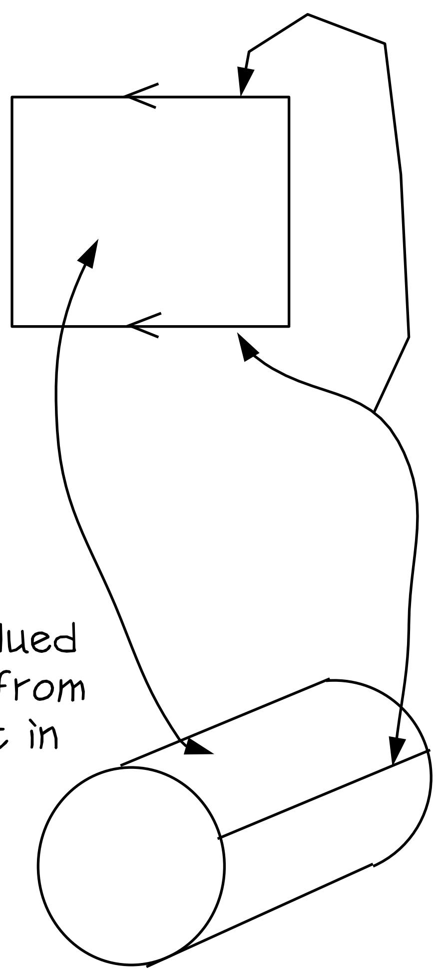
This givs a one-to-one and onto function f from R^2 to the plane



f is not just one-to-one and onto but is actually a homeomorphism

Identification Spaces

Intuition: If you glue opposite sides of a square you get a cylinder



The new space is composed from disjoint subsets of the original space

<u>Identification Spaces</u>

<u>Defn</u>: Let X be a topological space

Let P_α be a family of disjoint subsets of X such that

$\bigcup_\alpha P_\alpha = X$,

Let Y be a set whose points are members of P_α

Let p: X -> Y be the map that takes every point of X

to the subset containing Y

Let a subset O of Y be open if and only if $p^{-1}(O)$ is

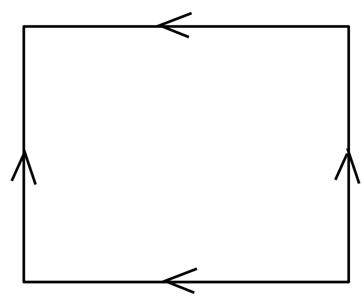
open in Y

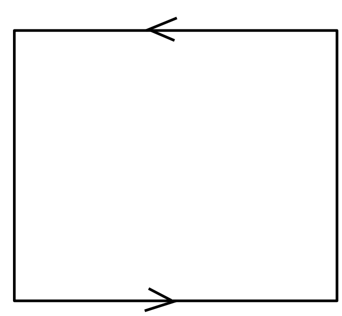
Under these conditions Y is a topological space with the

<u>identification topology</u>. (Y is an <u>identification</u>

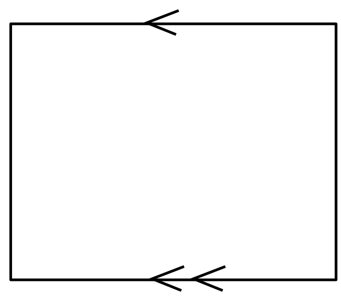
<u>space</u>)

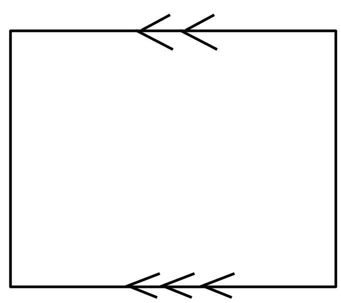
Identification Spaces

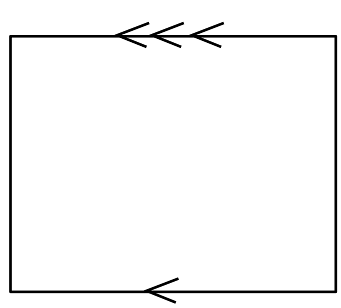




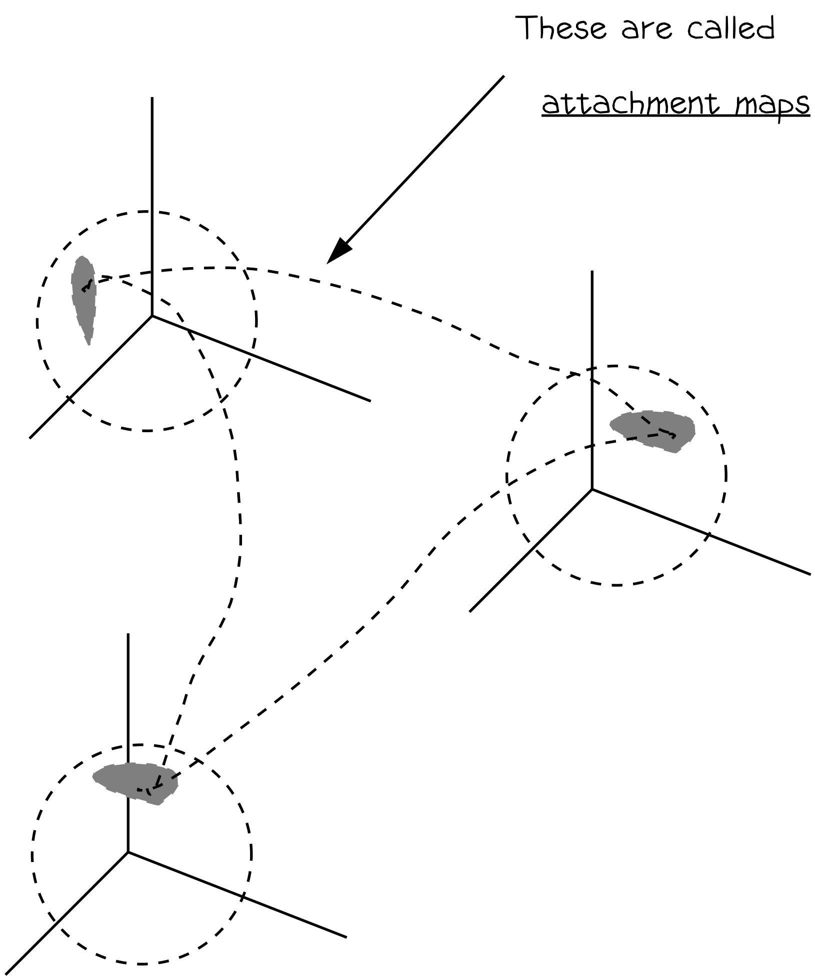
Identification Spaces







Identification Spaces



An extention of this idea gives us a

manifold

A really cool theorem

Let f: X -> Y be an onto and continuous function,

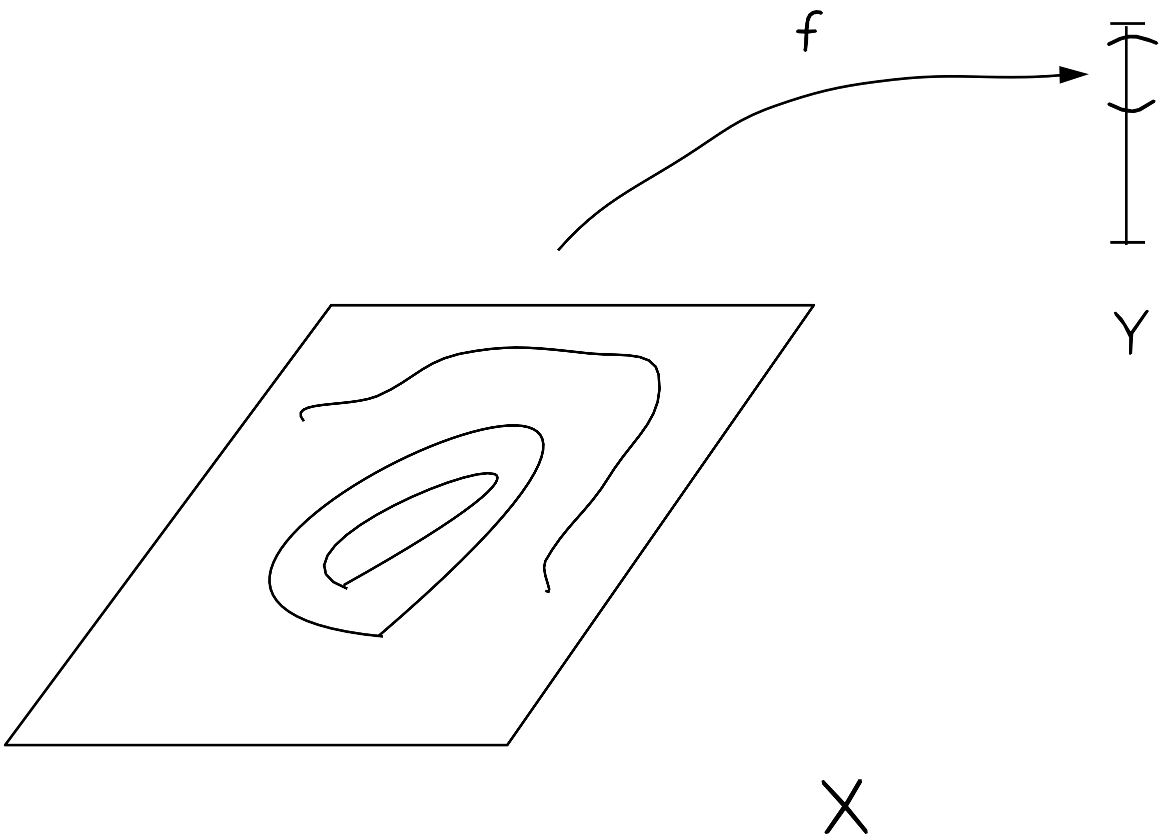
(and suppose that Y has the largest topology for which f is continuous), then f partitions X according to $f^{-1}(y)$, $\forall y \in Y$.

The technical condition is satisfied if X is compact and Y is Hausdorff

Let Y* be the identification space associated with the partition.

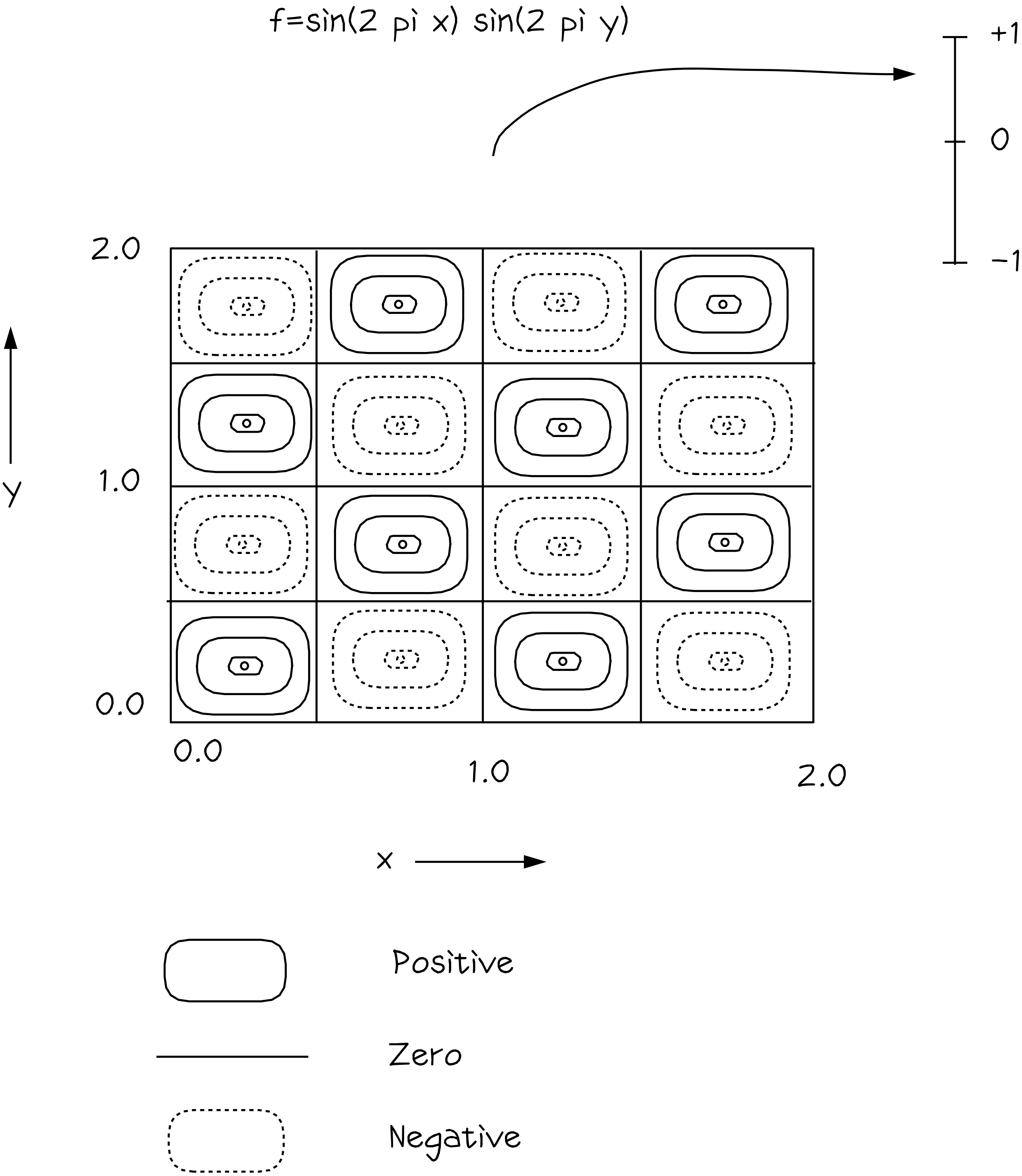
<u>Theorem</u>: Y* is homeomorphic to Y

Level Sets



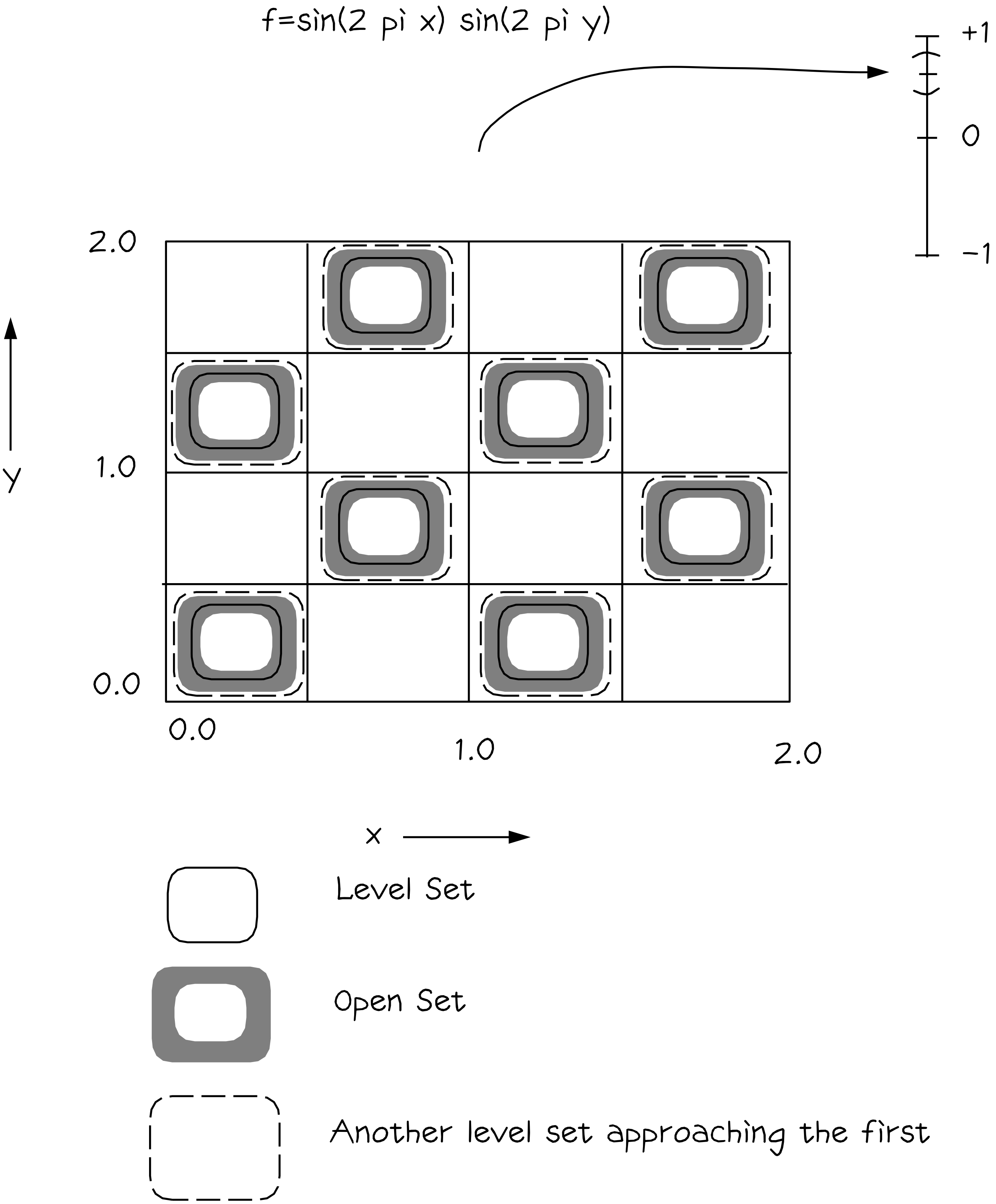
Level sets are connected exactly like R !!

Level Sets



f=sin(2 pi x) sin(2 pi y)
+1
0
−1
2.0
1.0
0.0
y
0.0
1.0
2.0
x
Positive
Zero
Negative

Level Sets



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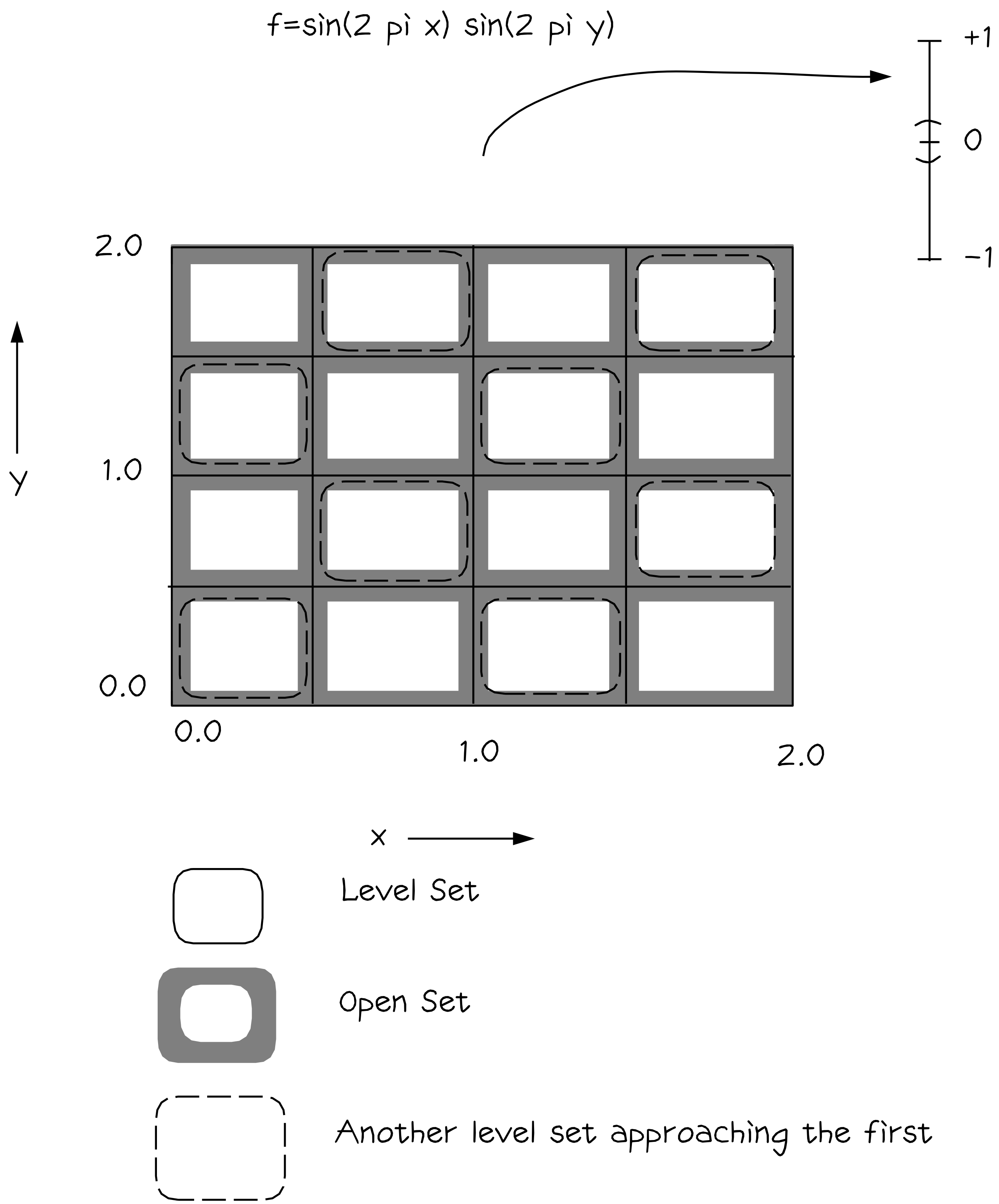
x
Level Set
Open Set
Another level set approaching the first

Level Sets



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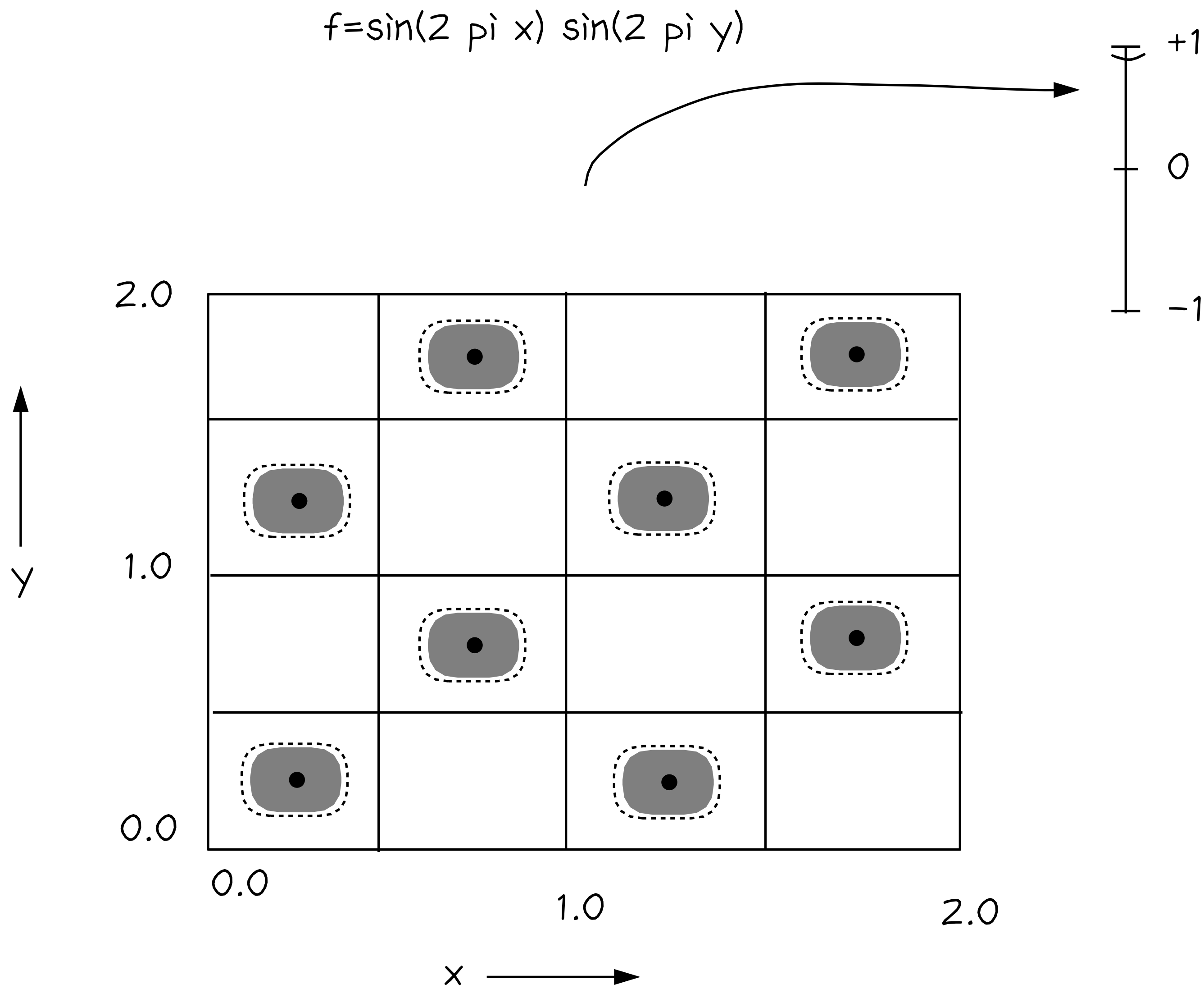

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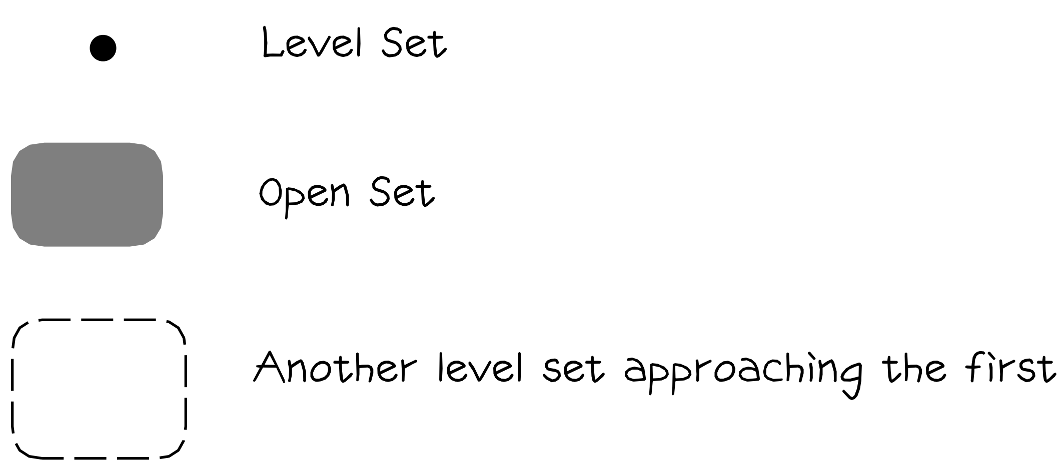

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Further Topology

Just as we generalized the notions of open sets and continuous functions, we can generlize the notions of connected and compact sets

Connectivity is a topological invariant

Invariants are important for showing when two topological spaces are not homeomorphic.

The key to doing all of this is to generalize common notions by using a set of formal properties.

surface
derivatives
vector

Knowing which properties to use takes genius.