Tangent Spaces in Plain English

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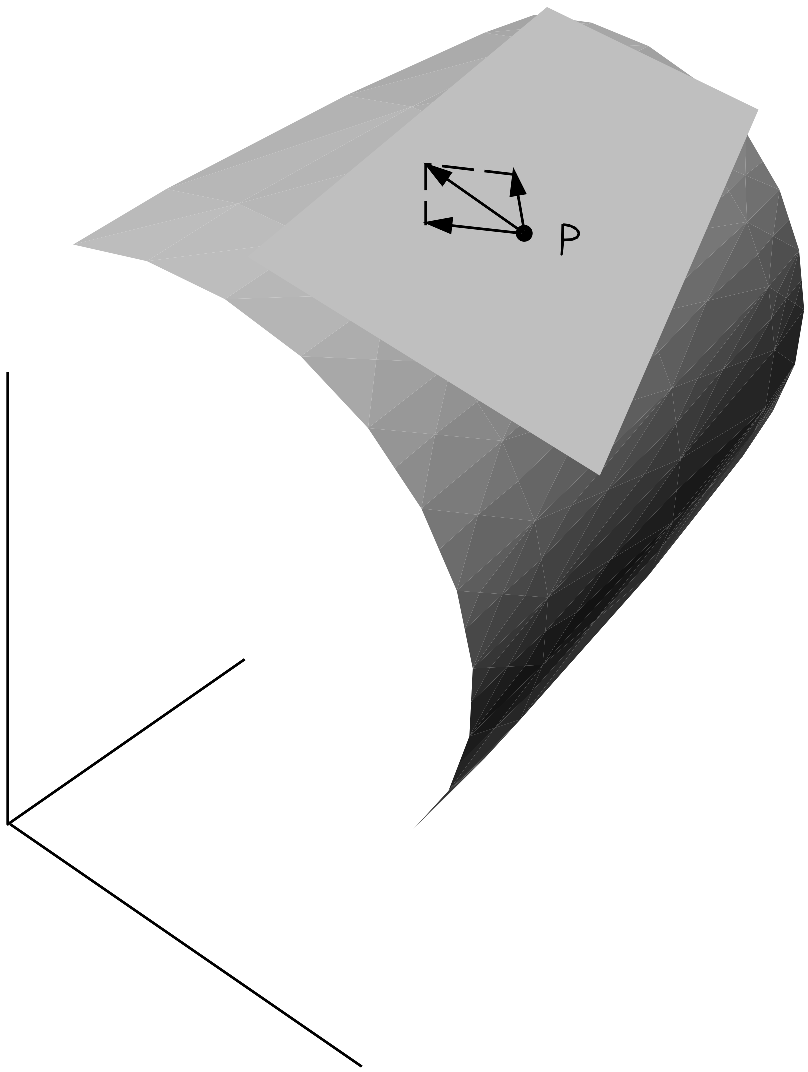
Tangent plane to a surface



The tangent plane at p is 2-D vector space with p as the origin

tangent space at p?

How do we generalize this?

Vector Space

Defn: A real vector space X is a set on which the operations of addition (+) and mutliplication (.) are well-defined.

That is

[a] $x + y = z$ is in X, whenenver x and y are in X
[b] $x + y = y + x$,
[c] $x + 0 = x$,

[d] $a.x$ is in x for any real number a and vector x

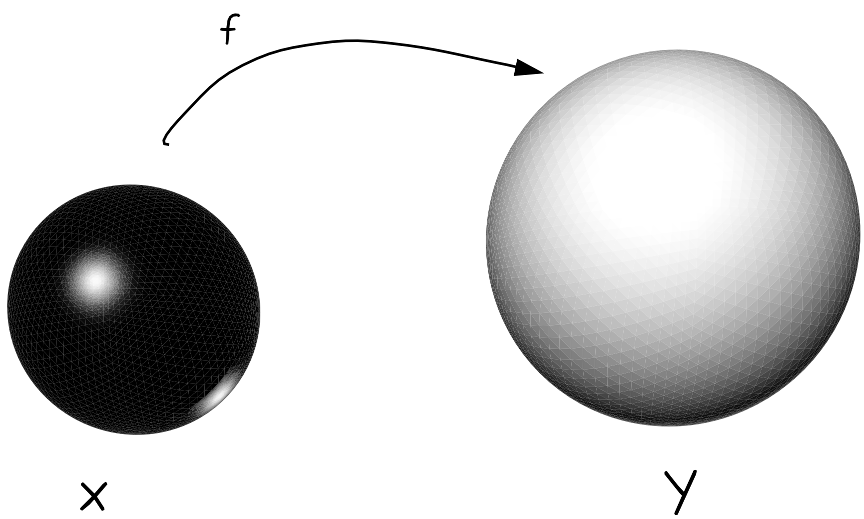
vector space -> basis -> dimension

linear
dependece

Theorem: If the number of elements in some basis set is finite then the number of elements in all basis sets (for a fixed space) is finite and fixed

The Euclidean n-space (R^n) is an example of an n-dimensional real-vector space.

Isomorphic Vector Spaces



Defn: Two vector spaces are isomorphic if there is a 1-1 mapping between them that conserves addition.

$$f(x +_X y) = f(x) +_Y f(y)$$

Isomorphic means "they are really the same space"

Theorem: Two real vector spaces are isomorphic if and only if they have the same dimension.

Theorem: All n-dimensional vector spaces are isomorphic to R^n

Isomorphic Vector Spaces

Theorem: All n-dimensional vector spaces are isomorphic to R^n

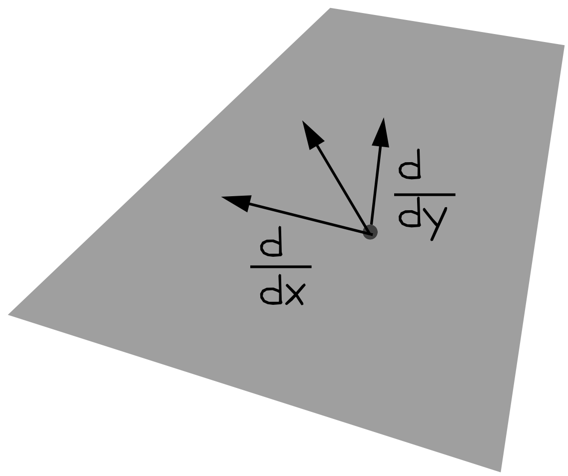
$$\begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

When we write a vector as what we are really writing is the real-valued co-efficients of the representation

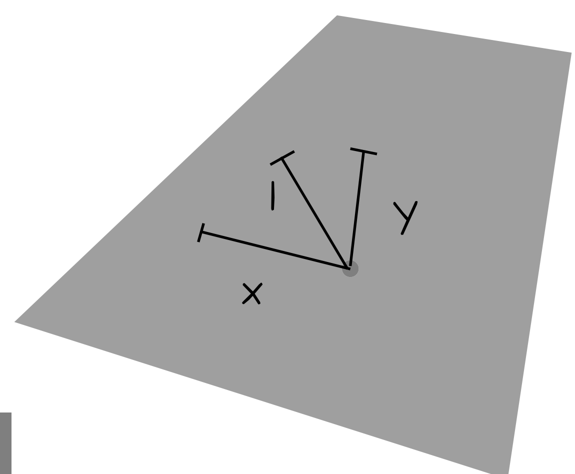
$$v = a_1 i_1 + a_2 i_2 + \ldots\ldots + a_n i_n$$

The basis vectors $i_1, i_2 \ldots\ldots$ do not have a numerical formula

Example

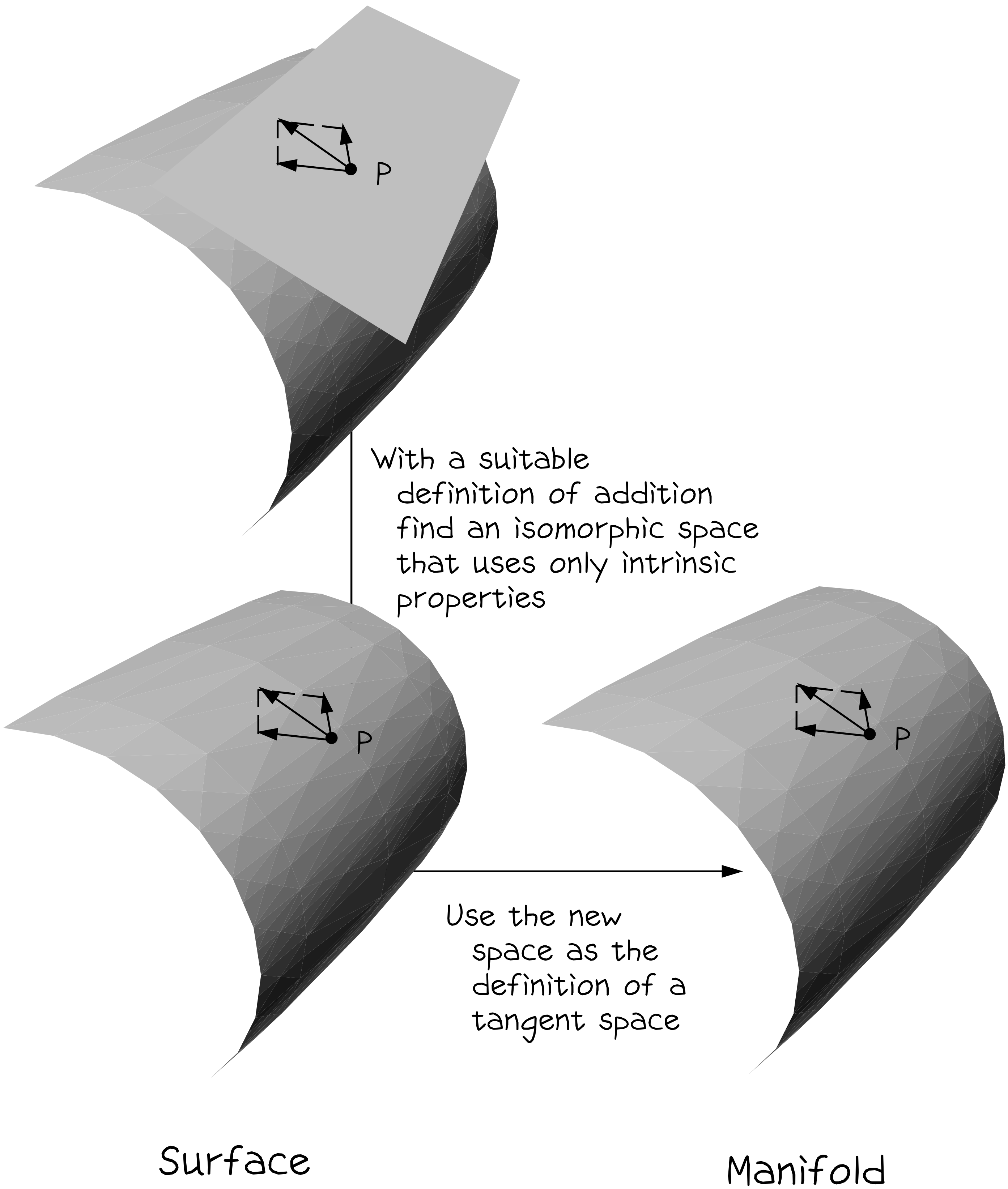


The set of all directional derivative operators at a point form a linear vector space

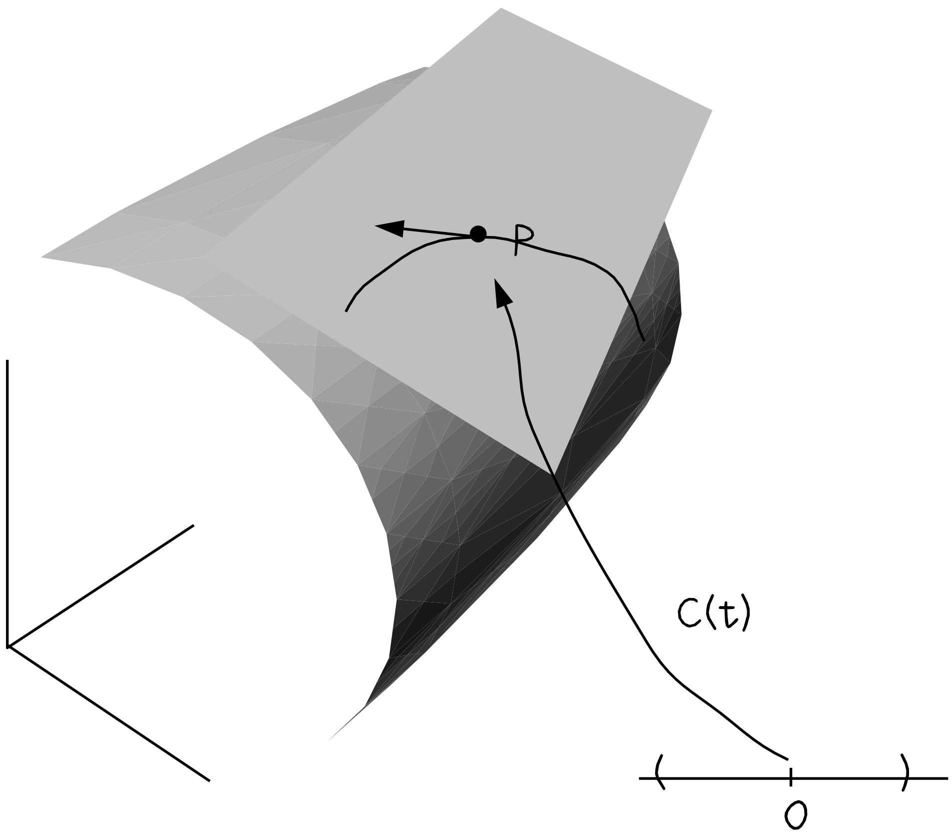


$$\frac{d}{dl} = a\frac{d}{dx} + b\frac{d}{dy}$$

Tangent plane to a surface



Observation 1



Let $C(t)$ be a curve on the surface passing through p for $t=0$

Then $\dfrac{dC(t)}{dt}$ is the tangent vector.

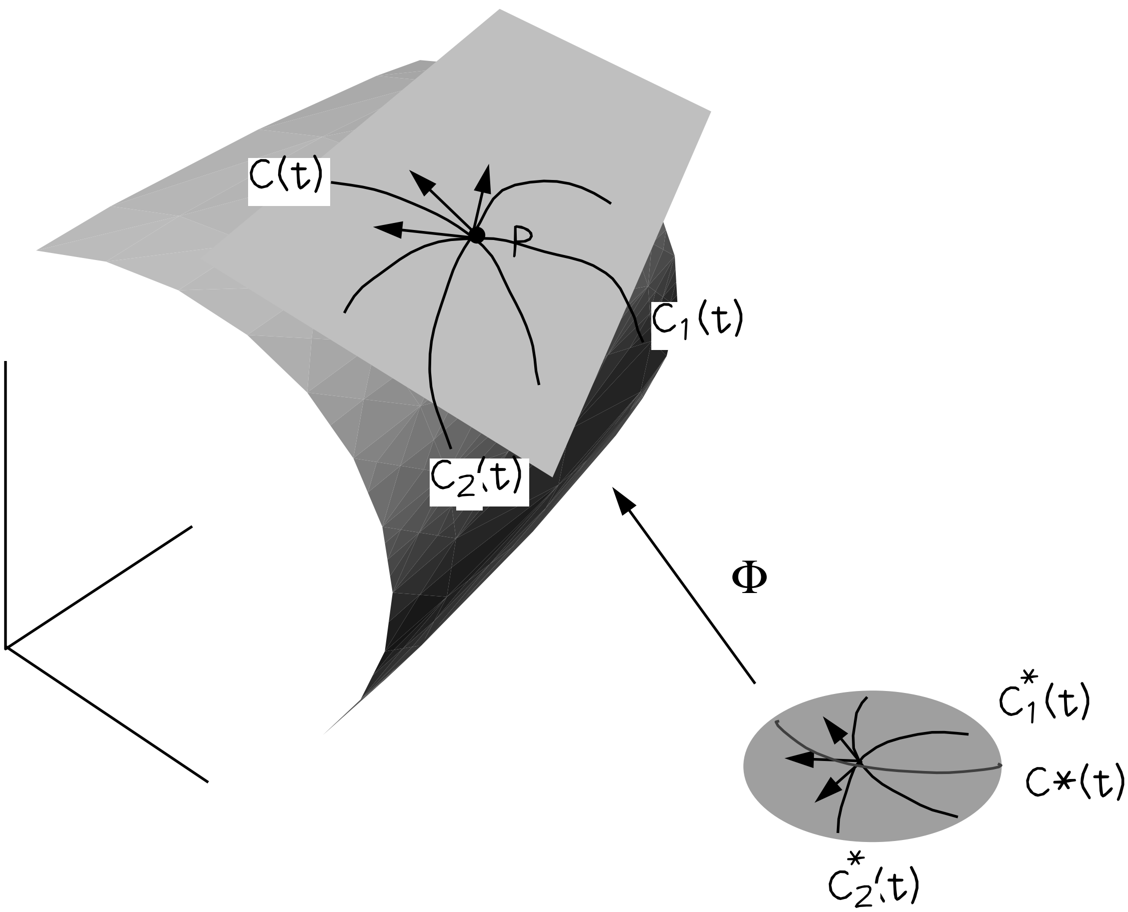
Multiplying the tangent vector by a means reparametrizing at a times the speed

Adding two tangent vectors means finding a curve such that its tangent vector at p is the (Euclidean) sum of the original tangent vectors. (NEEDS THEOREM)

The tangent plane is the vector space of tangents to curves at p.

Addition of tangent vectors

Theorem: The addition of tangent vectors is well-defined



Proof: $C_1(t) = \Phi(C^*_1(t))$, $\quad C_2(t) = \Phi(C^*_2(t))$

Let $C^*(t) = aC^*_1(t) + bC^*_2(t)$,

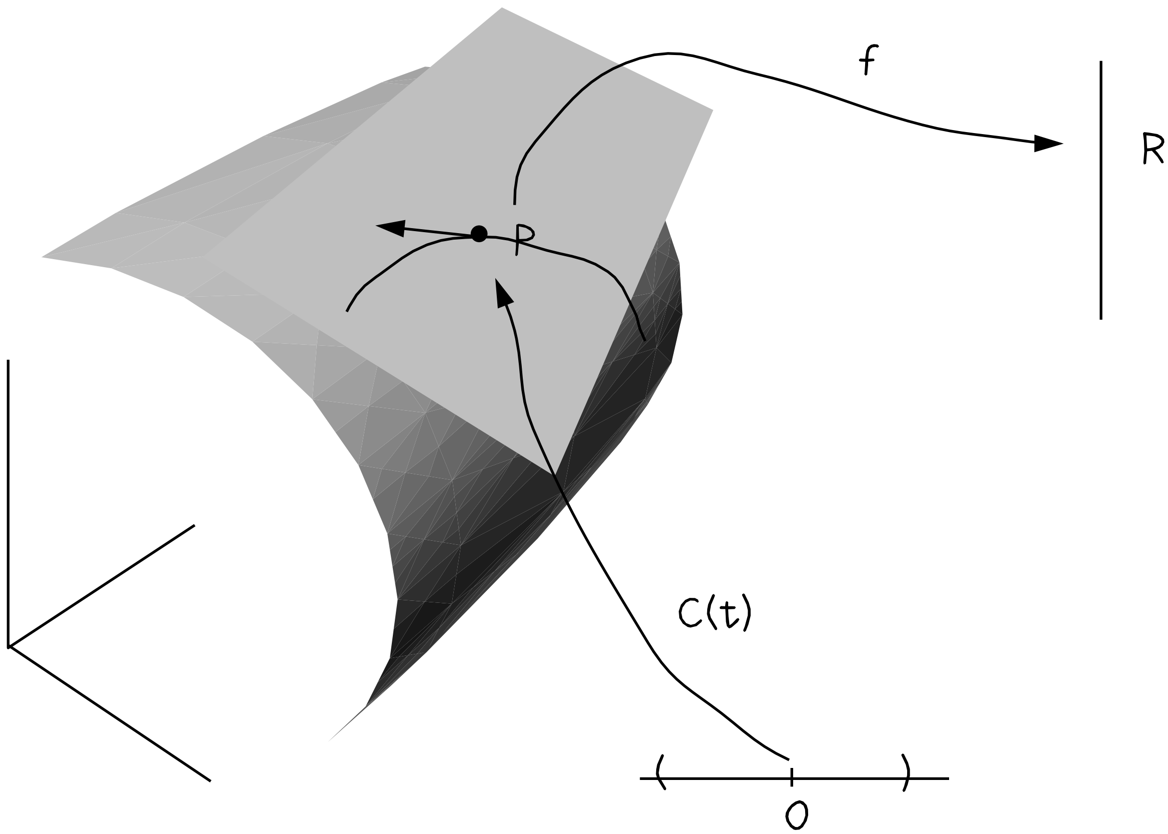
Then $\Phi(C^*(t))$ is a curve on the surface passing through point P. Call it $C(t)$.

$$\frac{dC(t)}{dt} = d\Phi \frac{dC^*(t)}{dt} = d\Phi \left(a \frac{dC^*_1(t)}{dt} + b \frac{dC^*_2(t)}{dt} \right)$$

$$= a\, d\Phi \frac{dC^*_1(t)}{dt} + b\, d\Phi \frac{dC^*_2(t)}{dt} = a \frac{dC_1(t)}{dt} + b \frac{dC_2(t)}{dt}$$

Observation 2 (Brilliant !! Herman Weyl, ~1930)

Tangent vectors also act as derivatives via the chain rule



f is a differentiable function from the surface to the real line

$f^*(t) = f(C(t)) \qquad (f^* = f \circ C)$

$$d_C f^*(t)\Big|_{t=0} = \frac{dC_x(t)}{dt}\frac{df}{dx} + \frac{dC_y(t)}{dt}\frac{df}{dy} + \frac{dC_z(t)}{dt}\frac{df}{dz}\Bigg|_{t=0}$$

Components of tangent vector

Observation 2

Think of $d_C[\ ]$ as an operator associated with C that takes as input a differential function on the surface and produces as an output the derivative of the function at p.

$$d_C[f] = \left.\frac{d\, f(C(t))}{dt}\right|_{t=0}$$

Two operators $d_{C_1}[f]$ and $d_{C_2}[f]$ are equal if and only if they give the same output for <u>every</u> differentiable function f

Observation 2

The set of all d operators forms a vector space

with suitably defined addition and multiplication

$d_{C_1} + d_{C_2} = d_C$ where C has a tangent vector which is the sum of tangent vectors of C_1 and C_2

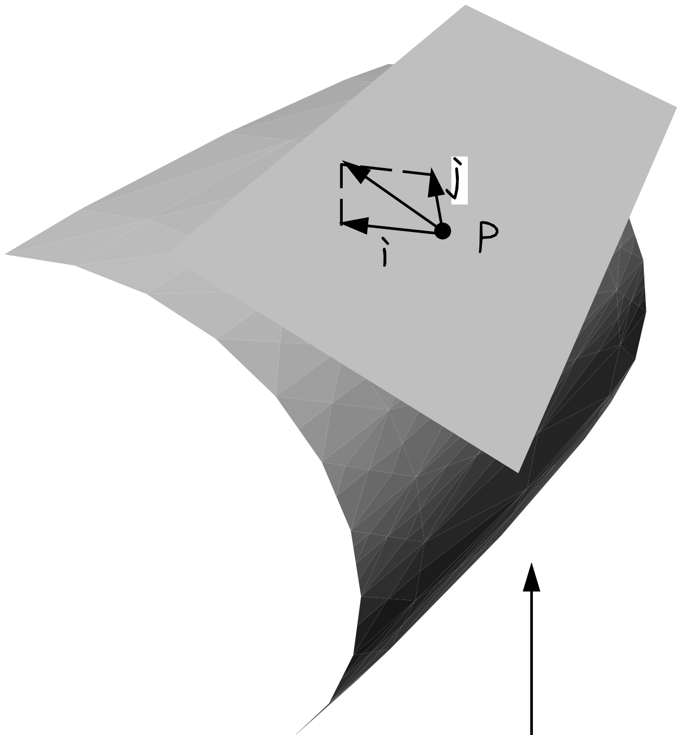
$a d_C = d_{Ca}$

Any d operator can be written as

$a d_{C_1} + b d_{C_2}$ for linearly independent operators d_{C_1} and d_{C_2}.

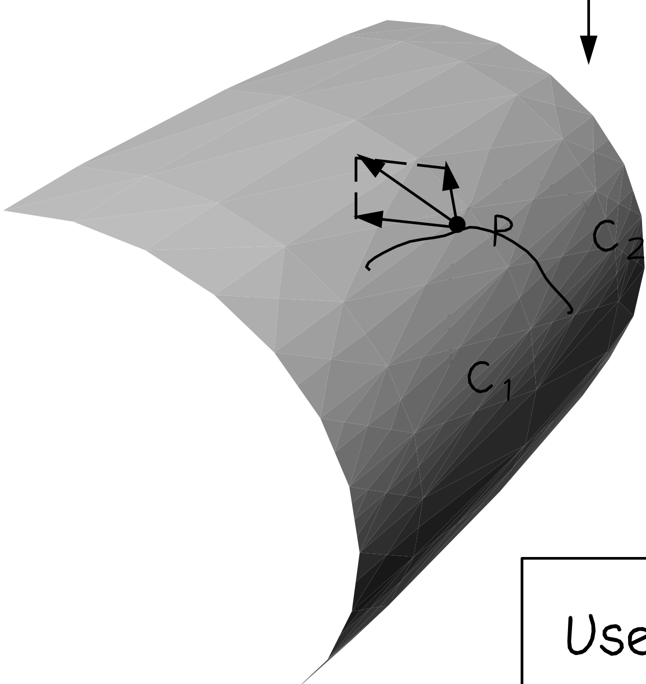
The space of d operators is isomorphic to the tangent space

Tangent Space and d-space



Every vector can
written as a linear
combination of i and j

Don't ask for a formula
for i and j



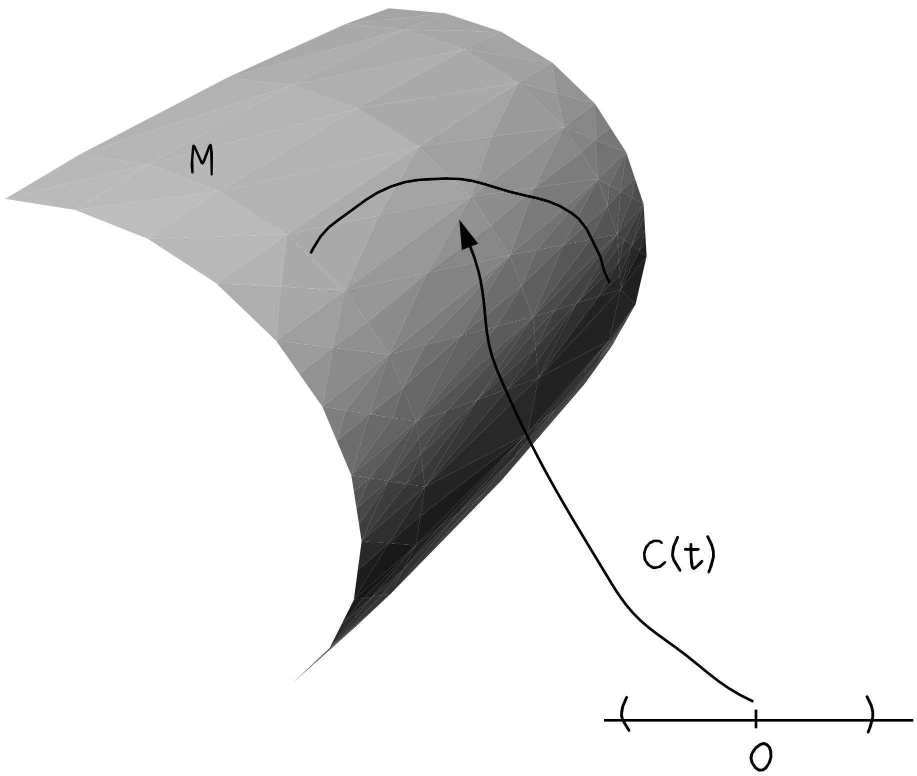
Every d_C operator can

as linear combinations
of d_{C_1} and d_{C_2} operators

Don't ask for a formula
for d_{C_1} and d_{C_2}

Use this definition

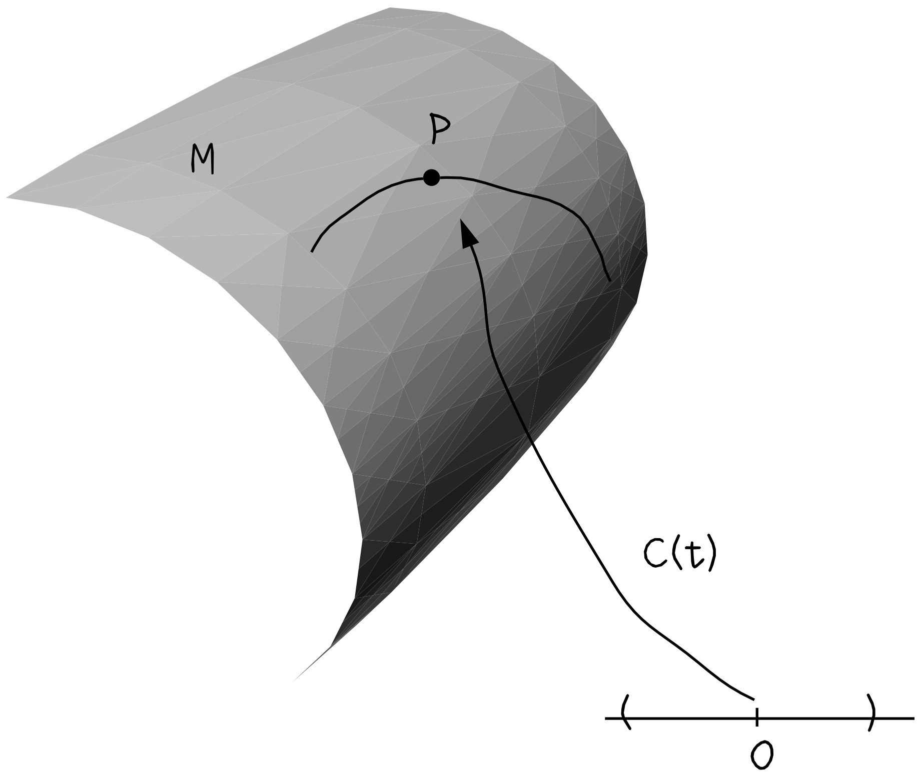
Curve on a Manifold



<u>Defn:</u> A curve on a manifold M is the image of a differentiable function C from (-d,d) to M.

<u>Note: No arc-lengths</u>

Curve on a Manifold

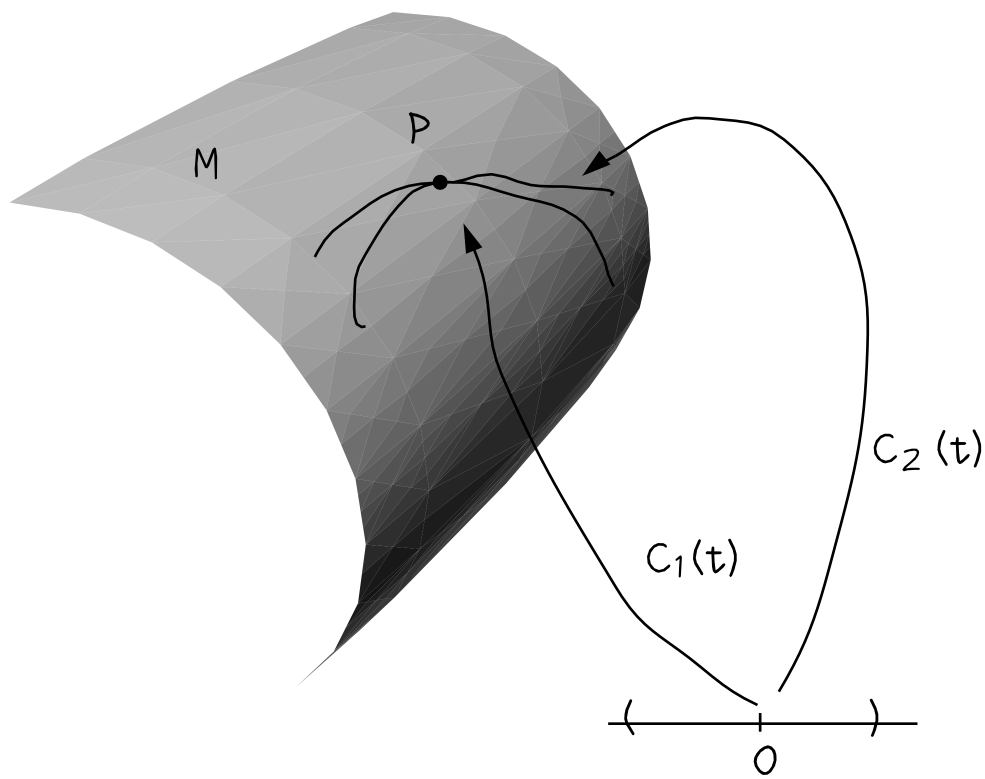


The curve passes through the point P if $C(0) = P$.

If f is a differentiable function on the manifold, then

the derivative of f w.r.t. C at P is $\frac{d\, f(C(t))}{dt}$ at $t=0$

Tangent vector at P



Defn: Two curves are tangent at P if for every f

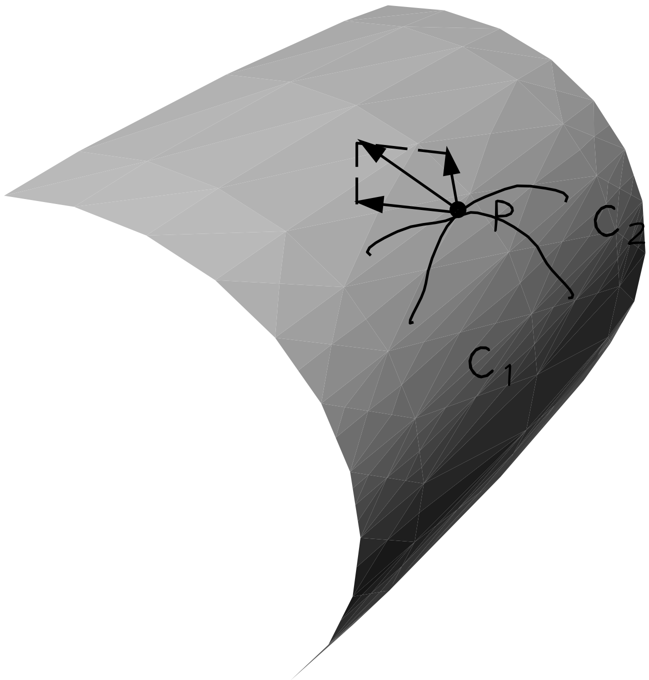
$$\frac{df(C_1(t))}{dt} = \frac{df(C_2(t))}{dt}$$

Note: No arc-lengths

Defn: The set of all curves tangent at P define a differential operator $v_{C_1} = v_{C_2} = \ldots\ldots$ called the tangent vector at P.

$v_C[f]$ gives the derivative of f along C ← Notation

Tangent space at P

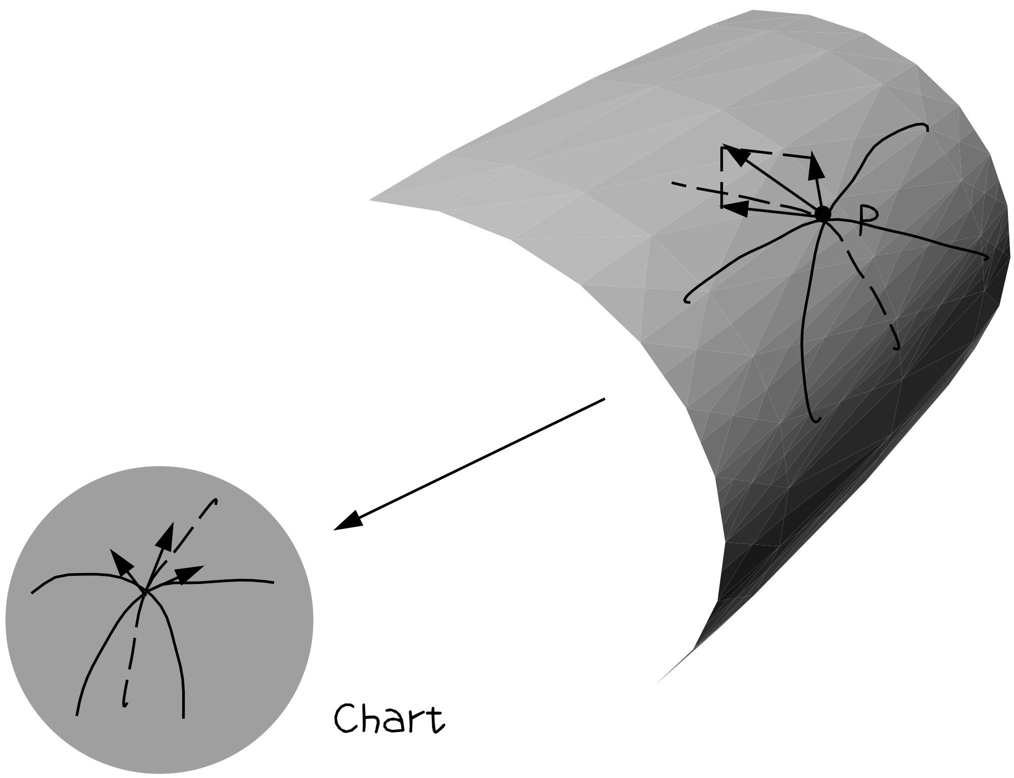


<u>Defn</u>: The set of all tangent vectors at P is the <u>tangent space</u> at P.

Convert it into a vector space by suitably (in the derivative sense) defining multiplication by scalar and addition

$$a\, v_C = v_{Ca}$$

Addition of tangent vectors



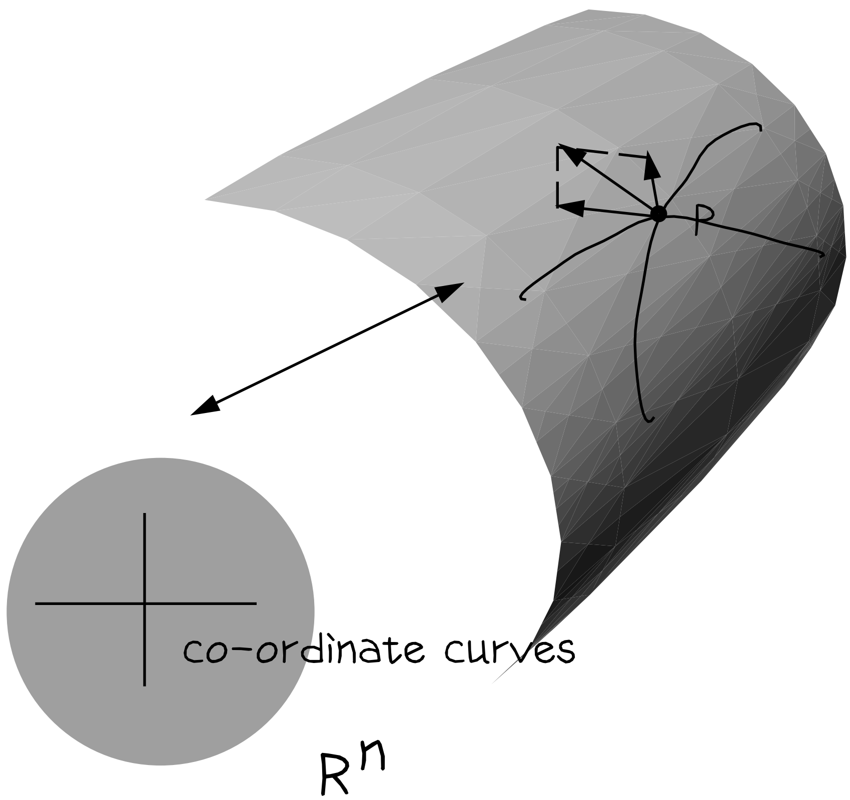
Chart

Theorem: $v_{C_1} + v_{C_2}$ is well defined

Theorem: The tangent space is a vector space

Theorem: The tangent space is iso-morphic to R^n for an n-dimensional manifold.

Standard Representation



Basis $\frac{dx_k}{dt}$. $k=1\ldots n$

Any tangent vector is written as

$$\sum_k a_k \frac{dx_k}{dt}$$