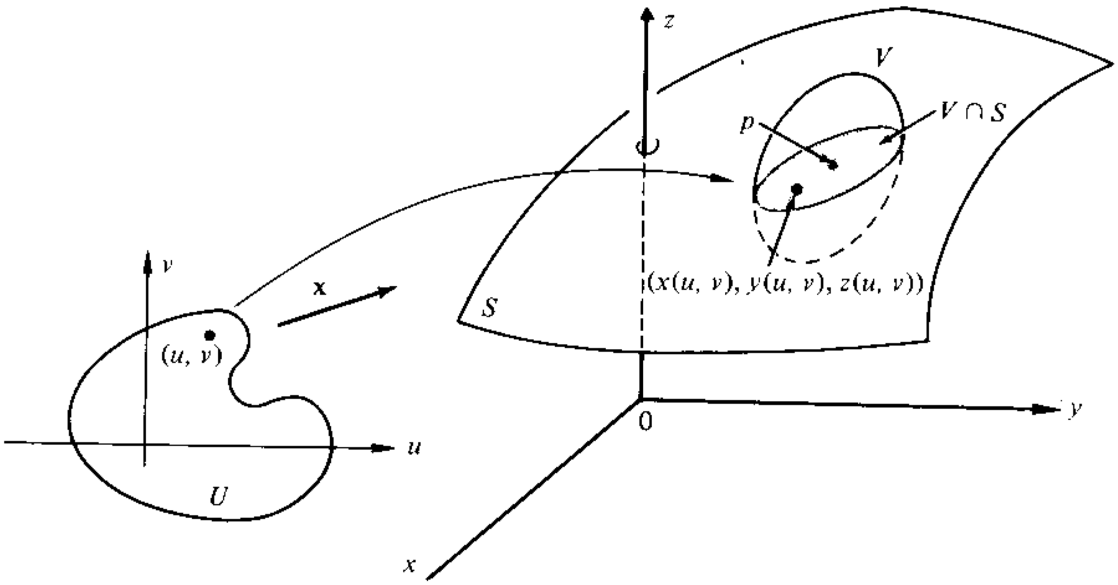
Regular Surfaces

Definition 1:

A subset $S \subset R^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in R^3 and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset R^2$ onto $V \cap S \subset R^3$ such that

- $\mathbf{x}$ is differentiable.
- $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is,$\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset R^3 \to R^2$ defined on an open set W containing $V \cap S$.
- For each $q \in U$, the differential $d\mathbf{x}_q : R^2 \to R^3$ is one-to-one. (*The regularity condition*).



The mapping $\mathbf{x}$ is called a *parameterization* or *a system of (local) coordinates* in (a neighborhood of) *p*. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood*.

Note that a surface is defined as a subset S of R^3, not as a map as in the curve case. This is achieved by covering S with the traces of parameterization which satisfy the three conditions.

Remarks:

- Condition 1 is natural if we need to do differential calculus on S.

- Condition 2 has the purpose of preventing self-intersection in regular surfaces. It is also essential to prove that certain objects defined in terms of a parameterization do not depend on this parameterization but only on S itself.

- Condition 3 (one of the Jacobian determinants do not equal to zero) will guarantee the existence of a tangent plane at all points of S.

Proposition 1:

If $f : U \to R$ is a differentiable function in an open set U of R^2, then the graph of f, that is, the subset of R^3 given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface.

Definition 2:

Given a differentiable map $F : U \subset R^n \rightarrow R^m$ defined in an open set U of R^n, we say that $p \in U$ is a *critical point* of F if the differential $dF_p : R^n \rightarrow R^m$ is not a surjective (or onto) mapping. The image $F(p) \in R^m$ of a critical point is called *critical value* of F. A point of R^m which is not a critical value is called a *regular value* of F.

$a \in f(U)$ is a regular value of $f : U \subset R^3 \rightarrow R$ if and only if f_x, f_y and f_z do not vanish simultaneously at any point in the inverse image

$$f^{-1}(a) = \{(x, y, z) \in U : f(x, y, z) = a)\}$$

Proposition 2:

If $f : U \subset R^3 \rightarrow R$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in R^3.

Proposition 3:

Let $S \subset R^3$ be a regular surface and $p \in S$. Then there exists a neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms: $z = f(x, y), y = g(x, z), x = h(y, z)$.

Proposition 1 says that the graph of a differentiable function is a regular surface. Proposition 3 provides a local converse of it; that is, any regular surface is locally the graph of a differentiable function.

Proposition 4:

Let $p \in S$ be a point of a regular surface S and let $\mathbf{x} : U \subset R^2 \to R^3$ be a map with $p \in \mathbf{x}(U) \subset S$ such that conditions 1 and 3 of Definition 1 hold. Assume that $\mathbf{x}$ is one-to-one, then $\mathbf{x}^{-1}$ is continuous.

It basically says that if we already know that S is a regular surface and we have a candidate $\mathbf{x}$ for a parameterization, we do not have to check that $\mathbf{x}^{-1}$ is continuous, provided that the other conditions hold.

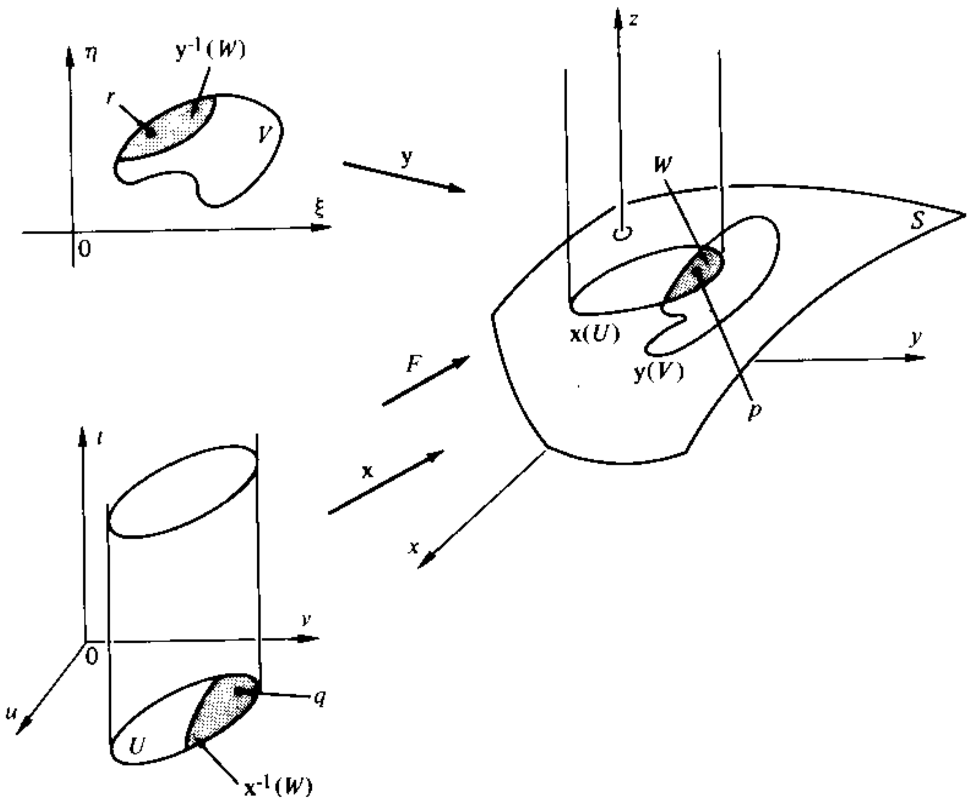
Change of Parameters

Remarks:

- We are interested in those properties of surfaces which depend on their behavior in a neighborhood of a point.

- For regular surfaces, each point p belongs to a coordinate neighborhood, and we should be able to define the local properties in terms of these coordinates.

- The same point p can, however, can belong to various coordinate neighborhoods. Moreover, other coordinate systems could be chosen in a neighborhood of p. It must be shown that when p belongs to two coordinate neighborhoods, it is possible to pass from one of the coordinates to the other by means of a differentiable transformation.



Proposition 1:

Let p be a point of a regular surface S, and let $\mathbf{x} : U \subset R^2 \to S, \mathbf{y} : V \subset R^2 \to S$ be two parameterizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the *change of coordinates* $h = \mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is a diffeomorphism; that is, h is differentiable and has a differentiable inverse h^{-1}.

If $\mathbf{x}$ and $\mathbf{y}$ are given by

$$\begin{aligned}\mathbf{x}(u,v) &= (x(u,v), y(u,v), z(u,v)), (u,v) \in U \\ \mathbf{y}(\xi,\eta) &= (x(\xi,\eta), y(\xi,\eta), z(\xi,\eta)), (\xi,\eta) \in V\end{aligned}$$

then the change of coordinate h, given by

$$u = u(\xi,\eta), v = v(\xi,\eta), (\xi,\eta) \in \mathbf{y}^{-1}(W)$$

has the property that the functions u and v have continuous partial derivatives of all orders, and the map h can be inverted, yielding

$$\xi = \xi(u,v), \eta = \eta(u,v), (u,v) \in \mathbf{x}^{-1}(W)$$

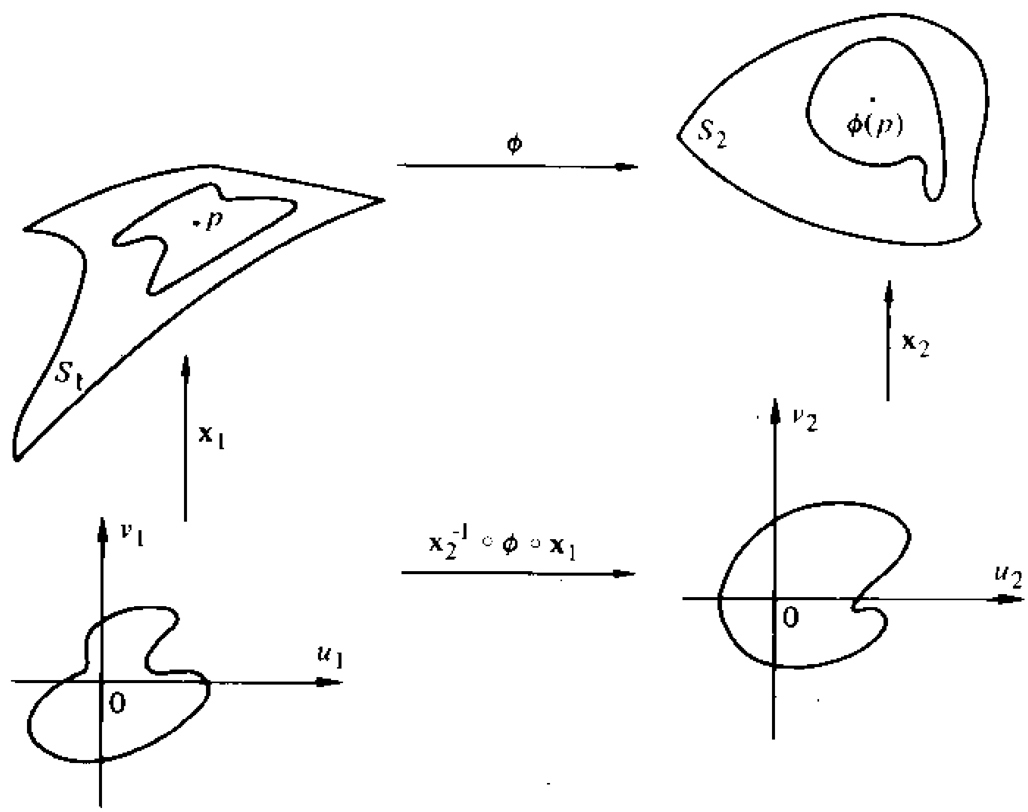
where the function ξ and η also have partial derivatives of all orders. Since

$$\frac{\partial(u,v)}{\partial(\xi,\eta)}\frac{\partial(\xi,\eta)}{\partial(u,v)} = 1$$

this implies that the Jacobian determinants of both h and h^{-1} are nonzero everywhere.

Definition 1:

Let $f : V \subset S \to R$ be a function defined in an open subset V of a regular surface S. Then f is said to be *differentiable* at $p \in V$ if, for some parameterization $\mathbf{x} : U \subset R^2 \to S$ with $p \in \mathbf{x}(U) \subset V$, the composition $f \circ \mathbf{x} : U \subset R^2 \to R$ is differentiable at $\mathbf{x}^{-1}(p)$. f is differentiable in V if it is *differentiable* at all points of V.



The definition of differentiability can be easily extended to mappings between surfaces. A continuous map $\varphi : V_1 \subset S_1 \to S_2$ of an open set V_1 of a regular surface S_1 to a regular surface S_2 is said to be differentiable at $p \in V_1$ if, given parameterizations

$$\mathbf{x}_1 : U_1 \subset R^2 \to S_1$$
$$\mathbf{x}_2 : U_2 \subset R^2 \to S_2$$

with $p \in \mathbf{x}_1(U)$ and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map

$$\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2$$

is differentiable at $q = \mathbf{x}_1^{-1}(p)$.

In other words, φ is differentiable if when expressed in local coordinates as

$$\varphi(u_1, v_1) = (\varphi_1(u_1, v_1), \varphi_2(u_1, v_1))$$

the functions φ_1, φ_2 have continuous partial derivatives of all orders.

Two regular surfaces S_1 and S_2 are *diffeomorphic* is there exists a differentiable map $\varphi : S_1 \to S_2$ with a differentiable inverse $\varphi^{-1} : S_2 \to S_1$. Such a φ is called a *diffeomorphism* from S_1 to S_2.