Parameterized Surfaces

Definition:

A parameterized surface $\mathbf{x} : U \subset R^2 \to R^3$ is a differentiable map $\mathbf{x}$ from an open set $U \subset R^2$ into R^3. The set $\mathbf{x}(U) \subset R^3$ is called the trace of $\mathbf{x}$.

$\mathbf{x}$ is regular if the differential $d\mathbf{x}_q : R^2 \to R^3$ is one-to-one for all $q \in U$ (i.e., the vectors $\partial\mathbf{x}/\partial u$, $\partial\mathbf{x}/\partial v$ are linearly independent for all $q \in U$). A point $p \in U$ where $d\mathbf{x}_p$ is not one-to-one is called a singular point of $\mathbf{x}$.

Proposition:

Let $\mathbf{x} : U \subset R^2 \to R^3$ be a regular parameterized surface and let $q \in U$. Then there exists a neighborhood V of q in R^2 such that $\mathbf{x}(V) \subset R^3$ is a regular surface.

Tangent Plane

Definition 1:

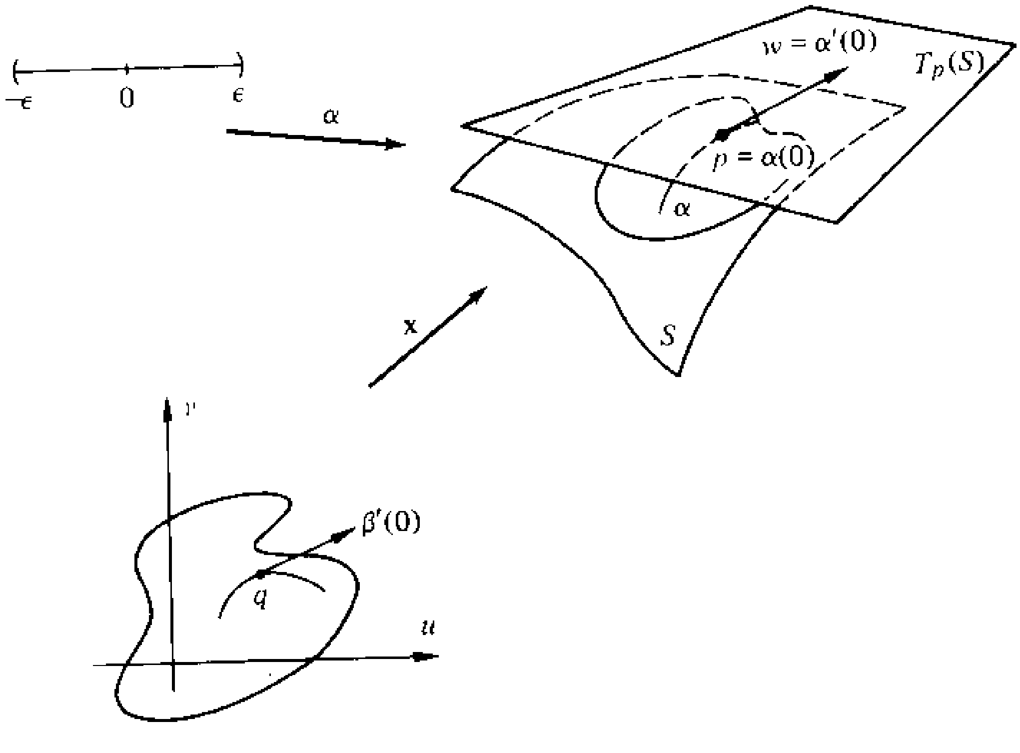
By a *tangent vector* to a regular surface S at a point $p \in S$, we mean the tangent vector $\alpha'(0)$ of a differentiable parameterized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

Proposition 1:

Let $\mathbf{x} : U \subset R^2 \to S$ be a parameterization of a regular surface S and let $q \in U$. The vector subspace of dimension 2,

$$d\mathbf{x}_q(R^2) \subset R^3$$

coincides with the set of tangent vectors to S at $\mathbf{x}(q)$.



Definition 2:

By Proposition 1, the plane $d\mathbf{x}_q(R^2)$, which passes through $\mathbf{x}(q) = p$, does not depend on the parameterization $\mathbf{x}$. This plane is called the *tangent plane* to S at p and will be denoted by $T_p(S)$.

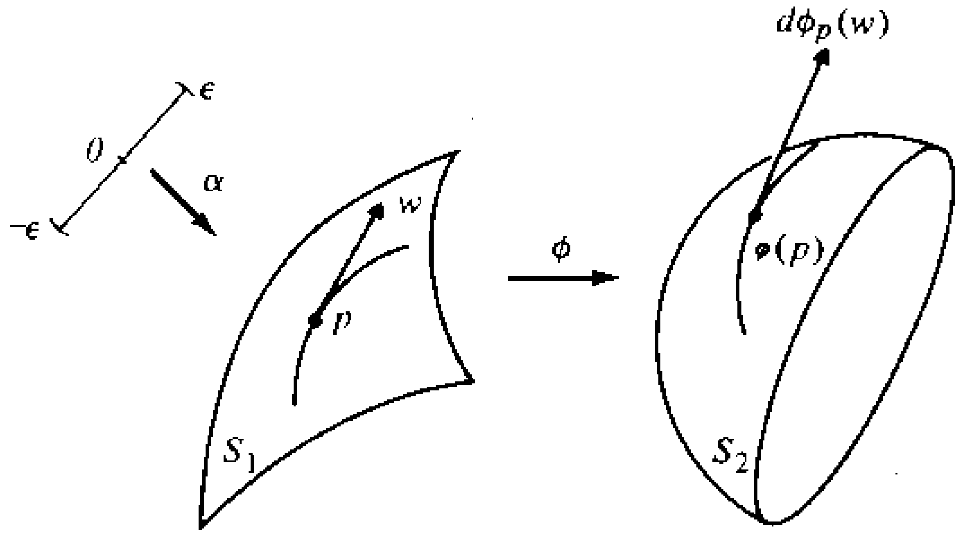
The choice of the parameterization $\mathbf{x}$ determines a basis $\{(\partial\mathbf{x}/\partial u)(q), (\partial\mathbf{x}/\partial v)(q)\}$ of $T_p(S)$, called the basis associated to $\mathbf{x}$.

The coordinates of a vector $w \in T_p(S)$ in the basis associated to a parameterization $\mathbf{x}$ are determined as follows:

w is the velocity vector $\alpha'(0)$ of a curve $\alpha = \mathbf{x} \circ \beta$, where $\beta : (-\epsilon, \epsilon) \to U$ is given by $\beta(t) = (u(t), v(t))$, with $\beta(0) = q = \mathbf{x}^{-1}(p)$. Thus,

$$\begin{aligned} \alpha'(0) &= \frac{d}{dt}(\mathbf{x} \circ \beta)(0) = \frac{d}{dt}\mathbf{x}(u(t), v(t))(0) \\ &= \mathbf{x}_u(q)u'(0) + \mathbf{x}_v(q)v'(0) \\ &= w \end{aligned}$$

Thus, in the basis $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$, w has coordinates $(u'(0), v'(0))$, where $(u(t), v(t))$ is the expression of a curve whose velocity vector at $t = 0$ is w.



Let S_1 and S_2 be two regular surfaces and let $\varphi : V \subset S_1 \to S_2$ be a differentiable mapping of an open set V of S_1 into S_2. If $p \in V$, then every tangent vector $w \in T_p(S_1)$ is the velocity vector $\alpha'(0)$ of a differentiable parameterized curve $\alpha : (-\epsilon, \epsilon) \to V$ with $\alpha(0) = p$. The curve $\beta = \varphi \circ \alpha$ is such that $\beta(0) = \varphi(p)$, and therefore $\beta'(0)$ is a vector of $T_{\varphi(p)}(S_2)$.

Proposition 2:

In the discussion above, given w, the vector $\beta'(0)$ does not depend on the choice of α. The map $d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ defined by $d\varphi_p(w) = \beta'(0)$ is linear.

This proposition shows that $\beta'(0)$ depends only on the map φ and the coordinates $(u'(0), v'(0))$ of w in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.

The linear map $d\varphi_p$ is called the *differential* of φ at $p \in S_1$. In a similar way, we can define the differential of a differentiable function $f : U \subset S \to R$ at $p \in U$ as a linear map $df_p : T_p(S) \to R$.

Proposition 3:

If S_1 and S_2 are regular surfaces and $\varphi : U \subset S_1 \to S_2$ is a differentiable mapping of an open set $U \subset S_1$ such that the differential $d\varphi_p$ of φ at $p \in U$ is an isomorphism, then φ is a local diffeomorphism at p.

The First Fundamental Form

Definition 1:
The quadratic form $I_p(w) = < w, w >_p = |w|^2 \geq 0$ on $T_p(S)$ is called the *first fundamental form* of the regular surface $S \subset R^3$ at $p \in S$.

The first fundamental form is merely the expression of how the surface S inherits the natural inner product of R^3. And by knowing I_p, we can treat metric questions on a regular surface without further references to the ambient space R^3.

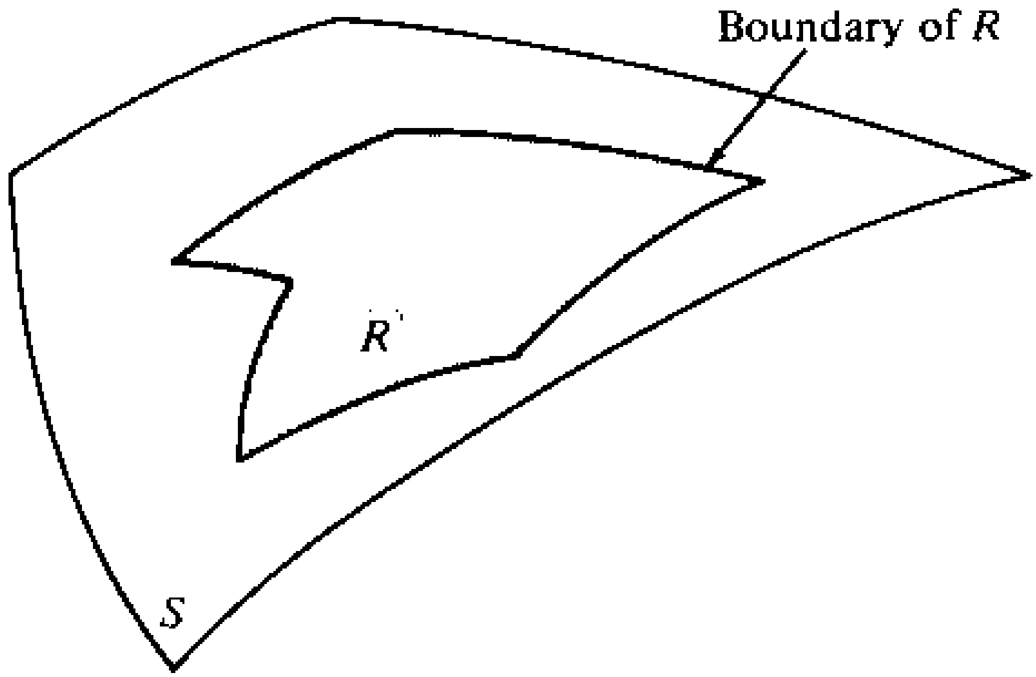
In the basis of $\{\mathbf{x}_u, \mathbf{x}_v\}$ associated to a parameterization $\mathbf{x}(u, v)$ at p, since a tangent vector $w \in T_p(S)$ is the tangent vector to a parameterized curve $\alpha(t) = \mathbf{x}(u(t), v(t)), t \in (-\epsilon, \epsilon)$, with $p = \alpha(0) = \mathbf{x}(u_0, v_0)$, we have

$$\begin{aligned}
I_p(\alpha'(0)) &= <\alpha'(0), \alpha'(0)>_p \\
&= <\mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v'>_p \\
&= <\mathbf{x}_u, \mathbf{x}_u>_p (u')^2 + 2<\mathbf{x}_u, \mathbf{x}_v>_p u'v' \\
&+ <\mathbf{x}_v, \mathbf{x}_v>_p (v')^2 \\
&= E(u')^2 + 2Fu'v' + G(v')^2
\end{aligned}$$

where the values of the functions involved are computed for $t = 0$, and

$$\begin{aligned}
E(u_0, v_0) &= <\mathbf{x}_u, \mathbf{x}_u>_p \\
F(u_0, v_0) &= <\mathbf{x}_u, \mathbf{x}_v>_p \\
G(u_0, v_0) &= <\mathbf{x}_v, \mathbf{x}_v>_p
\end{aligned}$$

are the coefficients.



Definition 2:

Let $R \subset S$ be a bounded region of a regular surface contained in the coordinate neighborhood of the parameterization $\mathbf{x} : U \subset R^2 \to S$. The positive number

$$A = \int\int |\mathbf{x}_u \times \mathbf{x}_v| \, dudv$$
$$= \int\int \sqrt{(EG - F^2)} \, dudv$$

is called the *area* of R.

Gauss Map

In the study of regular curve, the rate of change of the tangent line to a curve C leads to an important geometry entity, the curvature.

Here, we will try to measure how rapidly a surface S pulls away from the tangent plane $T_p(S)$ in a neighborhood of a point $p \in S$. This is equivalent to measuring the rate of change at p of a unit normal vector field N on a neighborhood of p, which is given by a linear map on $T_p(S)$.

Definition 1:

Given a parameterization $\mathbf{x} : U \subset R^2 \to S$ of a regular surface S at a point $p \in S$, a unit *normal vector* can be chosen at each point of $\mathbf{x}(U)$ by the rule

$$N(q) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}(q)$$

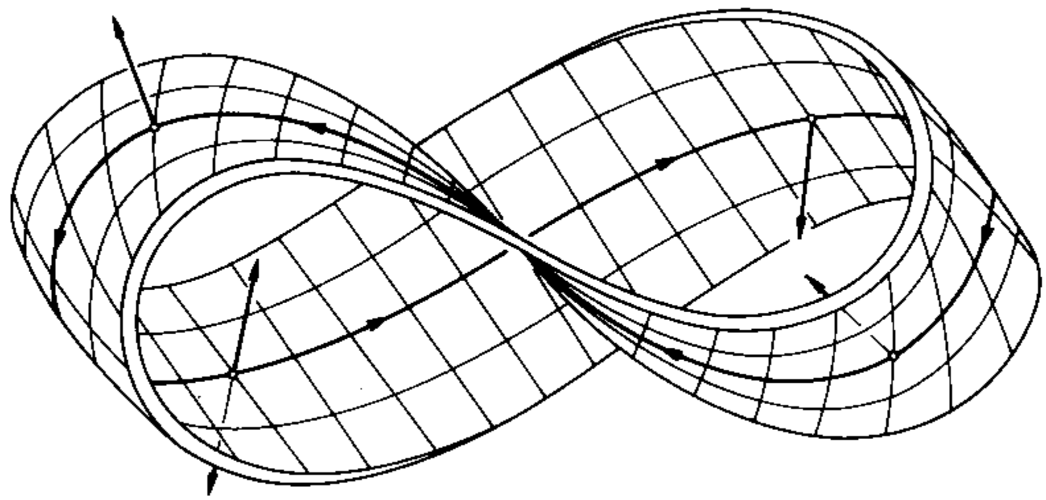
This way, we have a differentiable map $N : \mathbf{x}(U) \to R^3$ that associates to each $q \in \mathbf{x}(U)$ a unit normal vector $N(q)$.

More generally, if $V \subset S$ is an open set in S and $N : V \to R^3$ is a differentiable map which associates to each $q \in V$ a unit normal vector at q, we say that N is *a differentiable field of unit normal vectors on V*.

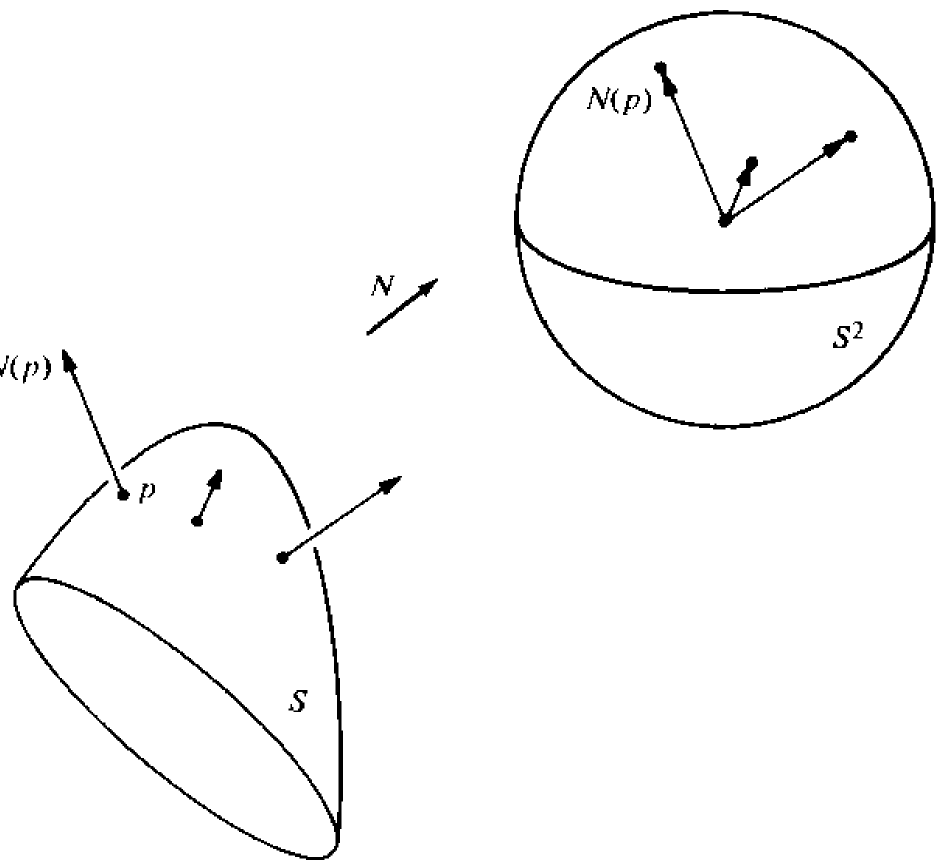
Definition 2:

A regular surface is *orientable* if it admits a differentiable field of unit normal vectors defined on the whole surface, and the choice of such a field N is called an *orientation* of S.

An orientation N on S induces an orientation on each tangent plane $T_p(S), p \in S$, as follows. Define a basis $\{v, w \in T_p(S)\}$ to be *positive* if $< v \times w, N >$ is positive.



While every surface is *locally orientable*, not all surfaces admit a differentiable field of unit normal vectors defined on the whole surface (i.e., the Mobius strip).

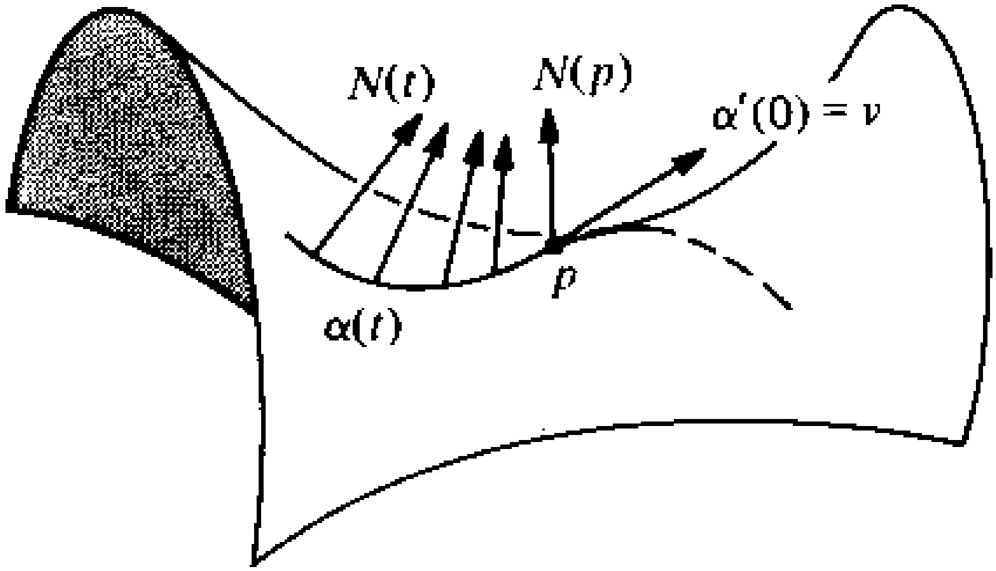


Definition 3:

Let $S \subset R^3$ be a surface with an orientation N. The map $N : S \to R^3$ takes its values in the unit sphere

$$S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$$

The map $N : S \to S^2$, thus defined, is called the *Gauss map* of S.



The linear map $dN_p : T_p(S) \to T_p(S)$ operates as follows. For each parameterized curve $\alpha(t)$ in S with $\alpha(0) = p$, we consider the parameterized curve $N \circ \alpha(t) = N(t)$ in the sphere S^2, this amounts to restricting the normal vector N to the curve $\alpha(t)$. The tangent vector $N'(0) = dN_P(\alpha'(0))$ is a vector in $T_p(S)$. It measures the rate of change of the normal vector N, restricted to the curve $\alpha(t)$, at $t = 0$. Thus, dN_p measures how N pulls away from $N(p)$ in a neighborhood of p.

Definition 4:

A linear map $A : V \to V$ is *self-adjoint* if $< Av, w >=< v, Aw >$ for all $v, w \in V$.

Proposition 1:

The differential $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self-adjoint linear map.

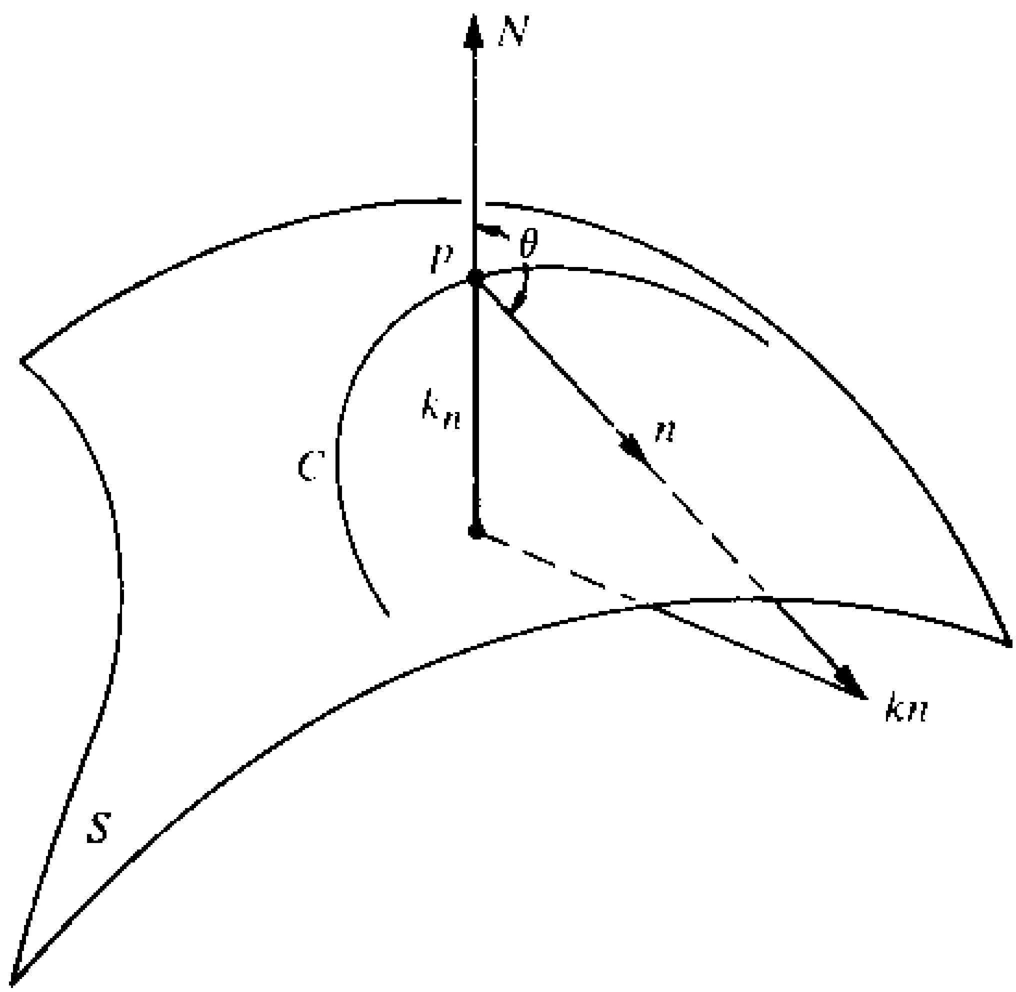
This proposition allows us to associate to dN_p a quadratic form Q in $T_p(S)$, given by $Q(v) =< dN_p(v), v >, v \in T_p(S)$.

Definition 5:

The quadratic form II_p, defined in $\in T_p(S)$ by $II_p(v) = - < dN_p(v), v >$, is called the *second fundamental form* of S at p.

Definition 6:

Let C be a regular curve in S passing through $p \in S$, k the curvature of C at p, and $cos\theta =< n, N >$, where n is the normal vector to C and N is the normal vector to S at p. The number $k_n = kcos\theta$ is then called the *normal curvature* of $C \ subsetS$ at p.


N
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P
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Consider a regular curve $C \subset S$ parameterized by $\alpha(s)$, where s is the arc length of C, and with $\alpha(0) = p$. If we denote by $N(s)$ the restriction of the normal vector N to the curve $\alpha(s)$, we have $< N(s), \alpha'(s) >= 0$. Hence,

$$< N(s), \alpha''(s) >= - < N'(s), \alpha(s) >$$

Therefore

$$\begin{aligned} II_p(\alpha'(0)) &= - < dN_p(\alpha'(0)), \alpha'(0) > \\ &= - < N'(0), \alpha'(0) > \\ &= < N(0), \alpha''(0) > \\ &= < N, kn > (p) \\ &= k_n(p) \end{aligned}$$

In other words, the value of the second fundamental form II_p for a unit vector $v \in T_p(S)$ is equal to the normal curvature of a regular curve passing through p and tangent to v.

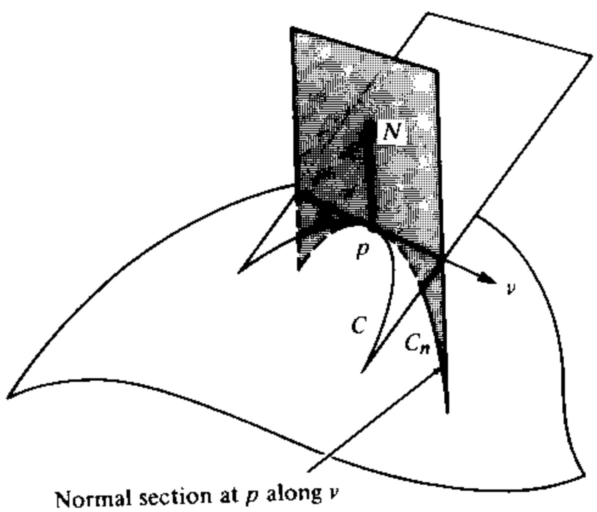

Figure 3-9. Meusnier theorem: C and C_n have the same normal curvature at p along v.

Proposition 2 (Meusnier Theorem:)
All curve lying on a surface S and having at a given point $p \in S$ the same tangent line have at this point the same normal curvature.