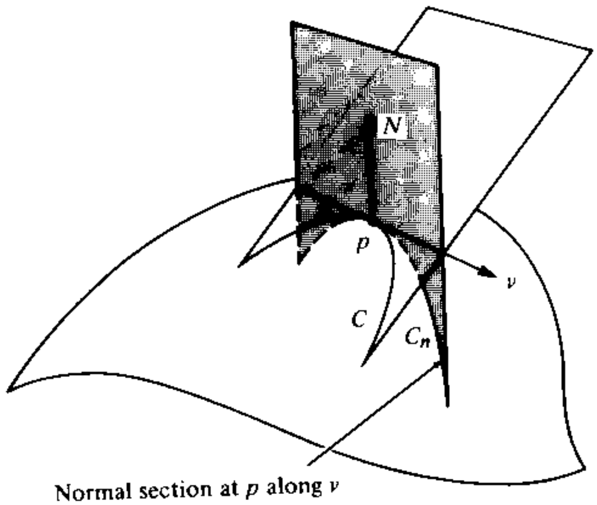


Figure 3-9. Meusnier theorem: C and C_n have the same normal curvature at p along v.

Proposition 2 (Meusnier Theorem:)

All curve lying on a surface S and having at a given point $p \in S$ the same tangent line have at this point the same normal curvature.

This proposition allows us to discuss the *normal curvature along a given direction at* p. Given a unit vector $v \in T_p(S)$, the intersection of S with the plane containing v and $N(p)$ is called the *normal section* of S at p along v.

Given a self-adjoint linear map $A : V \to V$, there exists an orthonormal basis for V such that relative to that basis the matrix of A is a diagonal matrix. Furthermore, the elements on the diagonal are the maximum and the minimum of the corresponding quadratic form restricted to the unit circle of V.

For each $p \in S$, there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$ such that $dN_p(e_1) = -k_1 e_1$, $dN_p(e_2) = -k_2 e_2$. Moreover, k_1 and k_2 ($k_1 \geq k_2$) are the maximum and minimum of the second fundamental for II_p restricted to the unit circle of $T_p(S)$; that is, they are the extreme values of the normal curvature at p.

Definition 7:

The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the *principal curvatures* at p; the corresponding directions, that is, the directions given by the eigenvectors e_1 and e_2, are called the *principal directions* at p.

The knowledge of the principal curvatures at p allows us to compute the normal curvature along a given direction of $T_p(S)$. Let $v \in T_p(S)$ with $|v| = 1$, and since e_1 and e_2 form an orthonormal basis of $T_p(S)$, we have

$$v = e_1 cos\theta + e_2 sin\theta$$

where θ is the angle from e_1 to v in the orientation of $T_p(S)$. The normal curvature k_n along v is given by the *Euler formula*:

$$\begin{aligned} k_n &= II_p(v) = -< dN_p(v), v > \\ &= -< dN_p(e_1 cos\theta + e_2 sin\theta), e_1 cos\theta + e_2 sin\theta > \\ &= < e_1 k_1 cos\theta + e_2 k_2 sin\theta, e_1 cos\theta + e_2 sin\theta > \\ &= k_1 cos^2\theta + k_2 sin^2\theta \end{aligned}$$

The Euler formula is just the expression of the second fundamental form in the basis $\{e_1, e_2\}$.

Definition 8:

Let $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinant of dN_p is the *Gaussian curvature* K of S at p. The negative of half of the trace of dN_p is called the *mean curvature* H of S at p.

In term of the principal curvatures, we have

$$\begin{aligned} K &= k_1 k_2 \\ H &= \frac{k_1 + k_2}{2} \end{aligned}$$

Definition 9:

A point of a surface S is called:

- *Elliptic*: if $det(dN_p) > 0$.
- *Hyperbolic*: if $det(dN_p) < 0$.
- *Parabolic*: if $det(dN_p) = 0$, with $dN_p \neq 0$.
- *Planar*: if $dN_p = 0$.

Definition 10:

If at $p \in S, k_1 = k_2$, then p is called an umbilical point of S; in particular, the planar points are umbilical points.

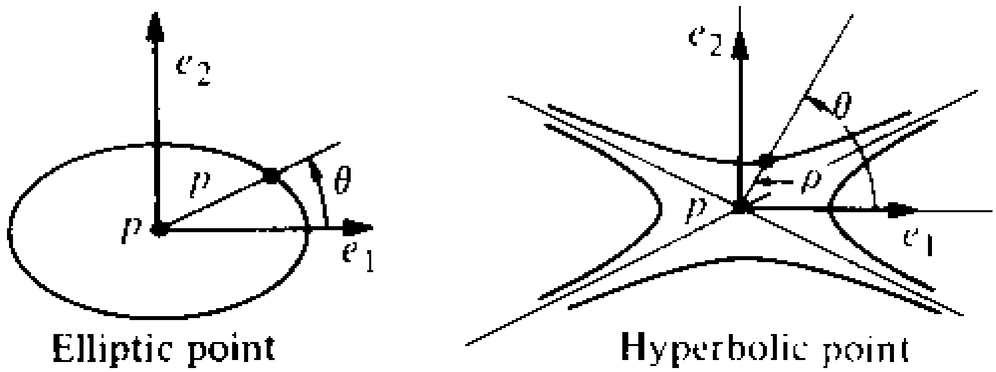
Definition 11 (Dupin indicatrix):

Let p be a point in S. The *Dupin indicatrix* at p is the set of vectors w of $T_p(S)$ such that $II_p(w) = \pm 1$.

In more convenient form, let (ξ, η) be the Cartesian coordinates of $T_p(S)$ in the orthonormal basis $\{e_1, e_2\}$, where e_1 and e_2 are eigenvectors of dN_p. Given $w \in T_p(S)$, let ρ and θ be the polar coordinates defined by $w = \rho v$, with $|v| = 1$ and $v = e_1 cos\theta + e_2 sin\theta$ if $\rho \neq 0$. By Euler formula,

$$\begin{aligned} \pm 1 = II_p(w) &= \rho^2 II_p(v) \\ &= k_1 \rho^2 cos^2\theta + k_2 \rho^2 sin^2\theta \\ &= k_1 \xi^2 + k_2 \eta^2 \end{aligned}$$

where $w = \xi e_1 + \eta e_2$. Hence, the Dupin indicatrix is a union of conics in $T_p(S)$.



- For an elliptic point, the Dupin indicatrix is an ellipse, and it degenerates into a circle if the point is an umbilical nonplanar point ($k_1 = k_2 \neq 0$).

- For a hyperbolic point, the Dupin indicatrix is made up of two hyperbolas with a common pair of asymptotic lines (zero normal curvature).

- For a parabolic point, the Dupin indicatrix degenerates into a pair of parallel lines.

Definition 12:

Let p be a point in S. An *asymptotic direction* of S at p is a direction of $T_p(S)$ for which the normal curvature is zero. An *asymptotic curve* of S is a regular connected curve $C \subset S$ such that for each $p \in C$ the tangent line of C at p is an asymptotic direction.

Definition 13:

Let p be a point in S. Two nonzero vectors $w_1, w_2 \in T_p(S)$ are conjugate if $< dN_p(w_1), w_2 >=< w_1, dN_p(w_2) >= 0$. Two directions r_1, r_2 at p are conjugate if a pair of nonzero vectors w_1, w_2 parallel to r_1 and r_2, respectively, are conjugate.

Gauss Map in Local Coordinates

From here on, all parameterizations $\mathbf{x} : U \subset R^2 \to S$ are assumed to be compatible with the orientation N of S; that is, in $\mathbf{x}(U)$,

$$N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

Let $\mathbf{x}(u, v)$ be a parameterization at a point $p \in S$, and let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a parameterized curve in S, with $\alpha(0) = p$.

The tangent vector to $\alpha(t)$ at p is

$$\alpha' = \mathbf{x}_u u' + \mathbf{x}_v v'$$

and

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v'$$

Since N_u and N_v belong to $T_p(S)$, we may write

$$\begin{aligned} N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \\ N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v \end{aligned}$$

and therefore

$$dN(\alpha') = (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v$$

hence,

$$dN\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}$$

This shows that in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, dN is given by the matrix (a_{ij}) which is not necessarily symmetric, unless $\{\mathbf{x}_u, \mathbf{x}_v\}$ is an orthonormal basis.

In the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, the second fundamental form is given by

$$\begin{aligned} II_p(\alpha') &= -< dN(\alpha'), \alpha' > \\ &= -< N_u u' + N_v v', \mathbf{x}_u u' + \mathbf{x}_v v' > \\ &= e(u')^2 + 2fu'v' + g(v')^2 \end{aligned}$$

where, since $< N, \mathbf{x}_u > = < N, \mathbf{x}_v > = 0$,

$$\begin{aligned} e &= -< N_u, \mathbf{x}_u > = < N, \mathbf{x}_{uu} > \\ f &= -< N_v, \mathbf{x}_u > = < N, \mathbf{x}_{uv} > = -< N_u, \mathbf{x}_v > \\ g &= -< N_v, \mathbf{x}_v > = < N, \mathbf{x}_{vv} > \end{aligned}$$

Weingarten Mapping:

The matrix $[\beta] = (a_{ij})$ in the form

$$[\beta] = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

is called the *Weingarten mapping matrix* or the shape operator matrix of the surface. This matrix combines the first and second fundamental forms into one matrix, and determines surface shape by relating the intrinsic geometry of the surface to the Euclidean (extrinsic) geometry of the embedding space.

The Gaussian curvature of a surface can be obtained from the Weingarten mapping matrix as its determinant:

$$K = det[\beta] = \frac{eg - f^2}{EG - F^2}$$

And the mean curvature is similarly half of the trace of the Weingarten mapping matrix:

$$H = \frac{tr[\beta]}{2} = \frac{eG - 2fF + gE}{2(EG - F^2)}$$

Koenderink Shape Index:

The signs of the Gaussian, mean and principal curvatures are often used to determine basic surface types and singular points such as umbilical points. Furthermore, the numerical relationship between the two principal curvatures are used in more detailed classification of surfaces by Koenderink, where a shape index function is defined as

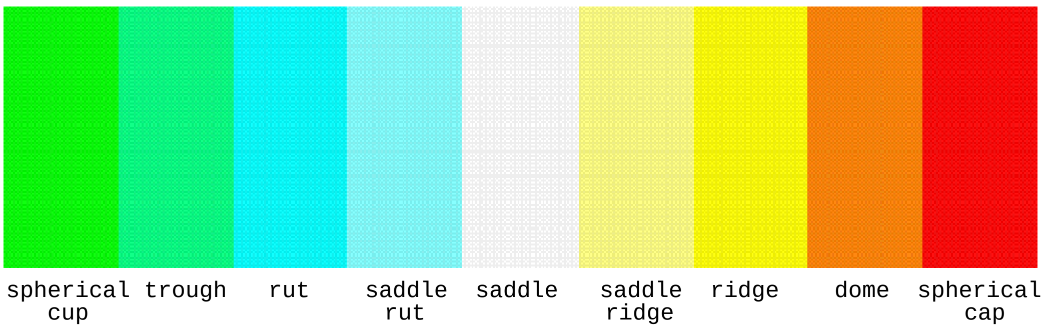
$$si = \frac{2}{\pi} arctan \frac{\kappa_2 + \kappa_1}{\kappa_2 - \kappa_1}, \ (\kappa_2 \geq \kappa_1)$$

This way, all surface patches, except for plane patches where the two principal curvatures equal zero, are mapped onto $si \in [-1, +1]$. This shape index function has many nice properties with regards to the classification of surface types:

- The shape index is scale invariant, i.e. two spherical patches with different radii will have same shape index values.

- Convexities and concavities are on the opposite sides of the shape index scale, separated by saddle-like shapes.

- Two shapes from which the shape index differs only in sign represent complementary pairs will fit to each other as stamp and mold if they are of same scale.



Surface Type	Shape Index Range
Spherical Cup	$si \in [-1, -7/8)$
Trough	$si \in [-7/8, -5/8)$
Rut	$si \in [-5/8, -3/8)$
Saddle Rut	$si \in [-3/8, -1/8)$
Saddle	$si \in [-1/8, +1/8)$
Saddle Ridge	$si \in [+1/8, +3/8)$
Ridge	$si \in [+3/8, +5/8)$
Dome	$si \in [+5/8, +7/8)$
Spherical Cap	$si \in [+7/8, +1]$

Shape Characterization of Discrete Surfaces

For a regular surface $S \subset R^3$ and $p \in S$, there always exists a neighborhood V of p in S such that V is the graph of a differentiable function which has one of the following three forms:

$$\begin{aligned} z &= h(x, y) \\ y &= s(x, z) \\ x &= t(y, z) \end{aligned}$$

Hence, given a point p of a surface S, we can choose the coordinate axis of R^3 such that the origin O of the coordinates is at p and the z axis is directed along the outward normal of S at p (thus, the xy plane agrees with the tangent plane $T_p(S)$). It follows that a neighborhood of p in S can be represented in the form $z = h(x, y), (x, y) \in U \subset R^2$, where U is an open set and h is a differentiable function, with $h(0, 0) = 0, h_x(0, 0) = 0, h_y(0, 0) = 0$.

In this case, the local surface is parameterized by

$$\mathbf{x}(u, v) = (u, v, h(u, v)), (u, v) \in U$$

where $u = x, v = y$. It can be shown that

$$\begin{aligned} \mathbf{x}_u &= (1, 0, h_u) \\ \mathbf{x}_v &= (0, 1, h_v) \\ \mathbf{x}_{uu} &= (0, 0, h_{uu}) \\ \mathbf{x}_{uv} &= (0, 0, h_{uv}) \\ \mathbf{x}_{vv} &= (0, 0, h_{vv}) \end{aligned}$$

Thus, the normal vector at (x, y) is

$$N(x, y) = \frac{(-h_x, -h_y, 1)}{(1 + h_x^2 + h_y^2)^{1/2}}$$

From the above expressions, it is easy to obtain the coefficients of the first and second fundamental forms as

$$
\begin{aligned}
E(x,y) &= 1 + h_x^2 \\
F(x,y) &= h_x h_y \\
G(x,y) &= 1 + h_y^2 \\
e(x,y) &= \frac{h_{xx}}{(1 + h_x^2 + h_y^2)^{1/2}} \\
f(x,y) &= \frac{h_{xy}}{(1 + h_x^2 + h_y^2)^{1/2}} \\
g(x,y) &= \frac{h_{yy}}{(1 + h_x^2 + h_y^2)^{1/2}}
\end{aligned}
$$

Hence, the curvatures of the surface can be derived from the Weingarten mapping matrix computed from these coefficients.