What is a Parametrization?

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1 Introduction

The goal of this note is to explain in some detail what we mean by "parametrization". We will use the idea of parametrization extensively, and it is useful to have a clear idea of what it is.

Definition: A set A *parameterizes* a set B if there is a one-to-one and onto function $p : A \to B$ ("onto" means that $p(A) = B$). We call the function p a *parameterizing function* (this is not a standard name). We will often choose a variable, say a to denote the elements of the set A. In that case, we call a the *parameter* of the set B. If the elements of A are numbers, then we have a *numerical parametrization.*

The picture we have in mind is that the parameterizing function takes every element in A and associates with it a unique element of B (fig. 1):

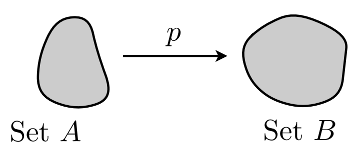

Figure 1: Set A parametrizes set B

As an example, let B be the set of functions on the unit interval $[0, 1]$ whose graphs are straight lines (figure 2), i.e. each element of B is a function (you should not think of this function as a formula but rather as a set theoretic concept we discussed in the first note). Let $A = \mathcal{R}^2$, so that each element of A is a pair of numbers, say (a, b). Let $p : A \to B$ map the element (a, b) to the function whose graph has slope and intercept a and b respectively. Clearly this is a one-to-one and onto function from A to B, and hence a parametrization of B.

The above is not the only way to parameterize B. Let $C = (-\pi/2, \pi/2) \times [0, \infty)$ be another set of ordered pairs of numbers (c, d) where c takes values in $(-\pi/2, \pi/2)$ and d takes values in $[0, \infty)$. Let $q : C \to B$ be the function that takes (c, d) to the function whose graph has a angle-displacement $c - d$ as shown in figure 3. Again, the function q is one-to-one and onto. Clearly, this is a second parametrization of B, but now with set C.

In this example, because the maps $p : A \to B$ and $q : C \to B$ are one-to-one and onto, the map $p^{-1} \circ q : C \to A$ is also one-to-one and onto (see fig 4). We can additionally think of $p^{-1} \circ q$ as

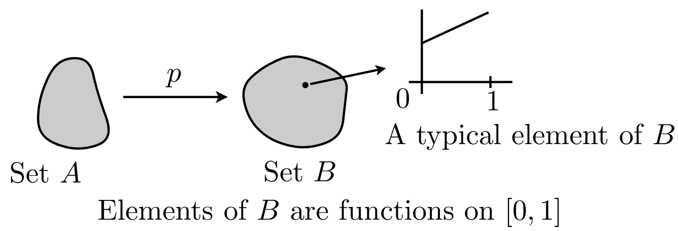

Figure 2: Parameterizing a set of functions

parameterizing the set A by the set C. Similarly, $q^{-1} \circ p : A \to C$ parameterizes the set C with the set A. We will call the maps $p^{-1} \circ q$ and $q^{-1} \circ p$ *change of parameter maps* or *reparametrization maps*.

Some additional standard terminology and comments:

1. If A is a finite (or countable) set that parameterizes B, then we say that B has a *discrete* parametrization.

   There is an alternate terminology for a discrete parametrization which you may be more familiar with. If A is a finite parametrization of B then we often say that A *indexes* B, and we use elements of A as subscripts to index elements of B.

2. If A is an open of set $\mathcal{R}^n$ that parameterizes B, then we say that B has a *continuous parametrization.*

3. If A and C are continuous parameterizations of the same set B, and the change of parameter maps are differentiable and have a full-rank Jacobian, then we say that the reparametrization is differentiable and full-rank. All reparametrizations that we consider will be differentiable and full-rank.

4. We are often faced with the following situation: We parameterize set B with set A and use calculations done with parameters (elements of set A) to claim that we have established something about elements of set B. For this claim to be strictly valid, we have to carefully check that the result really is independent of the parametrization we used. The claim is *not* valid if the result does not hold under reparametrization.

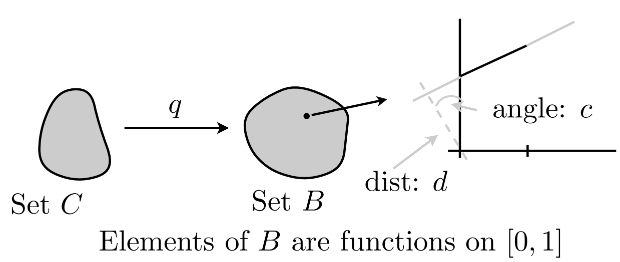

Figure 3: An alternate parametrization

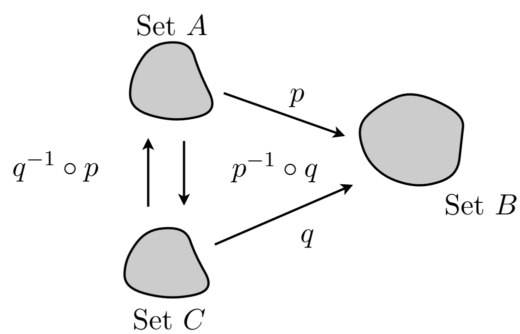

Figure 4: Reparametrization