

## HOW TO VISUALIZE SURFACES OF ROTATION

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**1. Why worry about visualization.** Many problems in integral calculus ask you to calculate the area of a surface generated by a curve rotating in three dimensions. The recipe for solving these problems goes like this :

1. First, you draw the curve. Then you mentally rotate it in space around an axis and imagine the surface generated by the rotation. This is the visualization part.
2. Based on the visualization, you write down an integral for the area of the surface.
3. Finally, you use various tricks you learn in the course to evaluate the integral.

The trouble is that most integral calculus courses assume you can visualize in 3-d, but many students get stuck at this step (the very first one in the recipe!). I am writing this tutorial for such students. If you have trouble visualizing surfaces of rotation, please read this tutorial carefully and do the exercises in the back of the tutorial. The answers are after the exercises.

**2. How to use this tutorial.** I am assuming that you have little or no facility with 3-d visualization. As you read the tutorial, you may find that you are more advanced. You will be tempted to skip some parts. They may even appear silly at first. Don't do it; don't skip. The latter sections depend explicitly on how I develop the material in earlier sections.

You should draw all figures by hand and draw them large. No grader likes to stare at a tiny figure. Above all, if you don't like how your figures look, keep practicing. They *will* get better.

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**3. Rotating a point in 3-D.** Let's start by visualizing how a point rotates in 3-D around a vertical axis. You should be sure that you can visualize all of the steps given here. Eventually you will not need to be so explicit, but when you start, you should develop this mental picture.

The figure below contains a vertical axis labelled *axis of rotation* and a point labelled 'a'. We want to rotate 'a' around the axis.

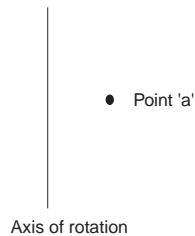


FIG. 3.1. *An axis and a point in 3-D.*

Let's do this in two steps:

**Step 1:** Find the point on the axis of rotation that is closest to 'a'. This is the point 'b' in the following figure. The closest point 'b' is always the one such that the line 'ab' is perpendicular to the axis of rotation

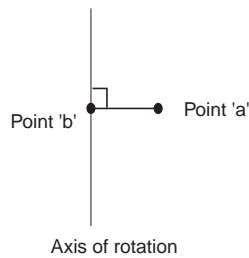


FIG. 3.2. *The closest point to 'a' on the axis of rotation.*

**Step 2:** Now imagine that the line 'ab' is a string that ties 'a' to the axis of rotation at 'b', and whirl the point 'a' around the axis:

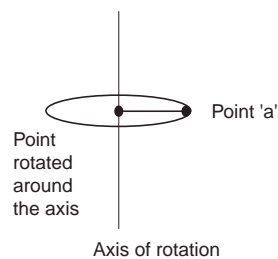


FIG. 3.3. *The point 'a' rotated around the axis.*

What you should see in your mind's eye is a circular ring or a halo around the axis of rotation.

**3.1. Common Pitfalls.** There are two common pitfalls you should be aware of:

1. The axis of rotation may not always be vertical. Nevertheless, the procedure is always the same. First find the closest point on the axis of rotation. Then draw a line from the point to be rotated to the closest point on the axis and whirl the point using this line. Here are some examples:

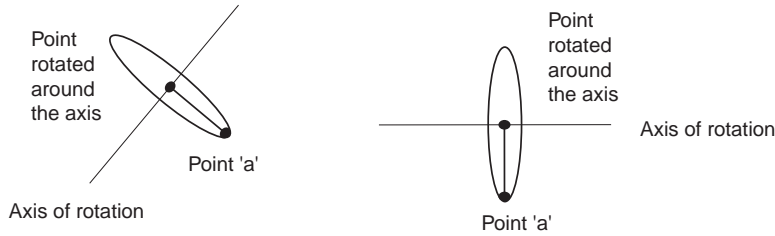


FIG. 3.4. *Non vertical axis.*

2. The point 'a' may lie on the axis. In this case the ring of rotation is the point itself:

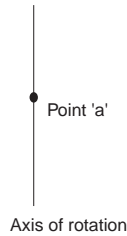


FIG. 3.5. *Point is on the axis.*

**4. Rotating multiple points.** Using the same procedure you can rotate multiple points around an axis of rotation. For each point to be rotated, find the closest point on the axis of rotation, draw the line to the point, and whirl it around the axis. Here I am rotating multiple points  $a_1, a_2, \dots$  around the axis:

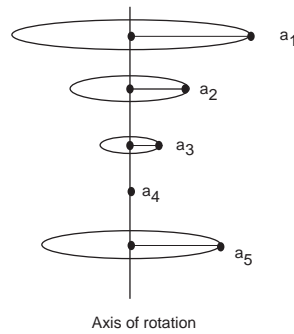


FIG. 4.1. *Rotating multiple points.*

Notice the following: The rotation of each point is a ring. The point  $a_4$  is on the axis of rotation, so its ring is just a point. All rings are parallel to each other and do not intersect.

The points that are to be rotated can lie anywhere in space. Here for example, they are on both sides of the axis:

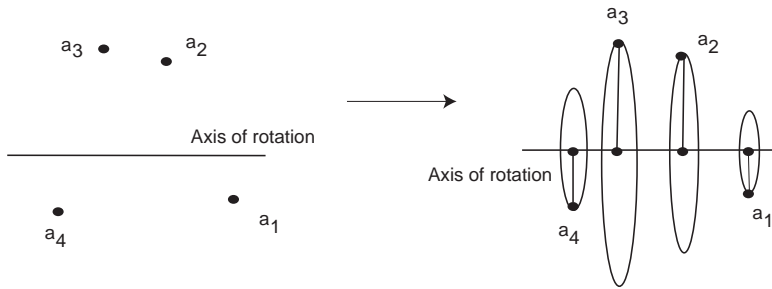


FIG. 4.2. *Points can be anywhere.*

**5. Rotating a curve in 3-D.** Next we will visualize how a curve rotates in 3-D. Here is a curve and an axis of rotation:

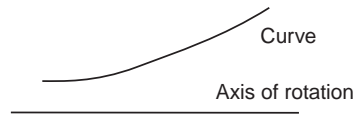


FIG. 5.1. *A curve and a rotation axis.*

Again we will do this in simple steps.

**Step 1:** Starting from one end of the curve, put down a set of points roughly at the same interval from each other. Mark them in sequence as  $a_1, a_2, \dots$ .

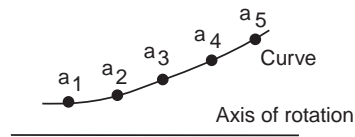


FIG. 5.2. *A curve and a rotation axis.*

**Step 2:** Ignoring the curve, just rotate the points around the axis with the techniques you have learnt so far:

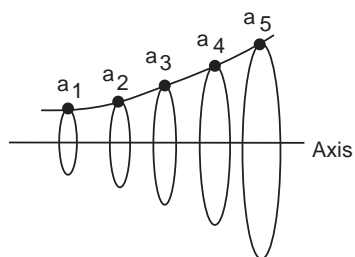


FIG. 5.3. Rotate the masked points.

**Step 3:** On each ring mark the points opposite to  $a_1, a_2, \dots$  as  $b_1, b_2, \dots$

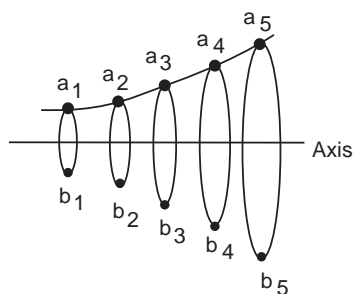


FIG. 5.4. Mark opposite points.

Be sure that you mark the points that are opposite to the original points, and be sure that you have named the points in the correct sequence. The point  $b_1$  must be on the same ring as  $a_1$  but opposite to it, the point  $b_2$  must be on the same ring as  $a_2$  but opposite to it, and so on.

**Step 4:** Next joint  $b_1$  to  $b_2$  to  $b_3 \dots$  in a smooth curve.

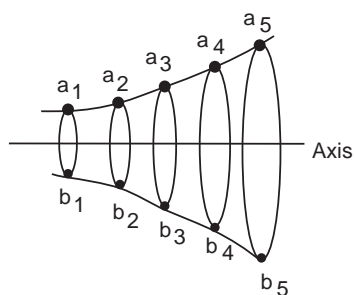


FIG. 5.5. Joint the marked points.

**Step 5:** Now imagine that the curve  $a_1, a_2, \dots$  slides around the rings in a complete circle. Note that this sliding curve passes through the curve  $b_1, b_2, \dots$ .

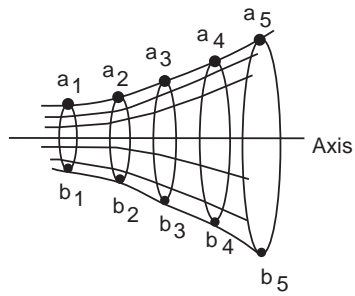


FIG. 5.6. *Joint the marked points.*

**Step 6:** Finally (and this may be little hard at first) erase (in your mind) all the construction we have done so far. That is, erase the points  $a_1, a_2, \dots$ , the rings of rotation, the points  $b_1, b_2, \dots$ . You should be able to see that as the curve whirls around the rotation axis, it creates a surface that looks like a water glass turned horizontal.

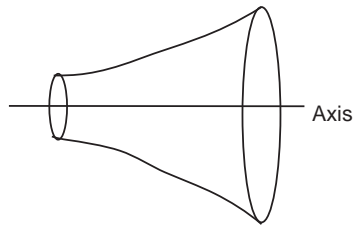


FIG. 5.7. *The final surface.*

**5.1. Another Example.** Just to illustrate this again, I am going to rotate the same curve around a vertical axis:

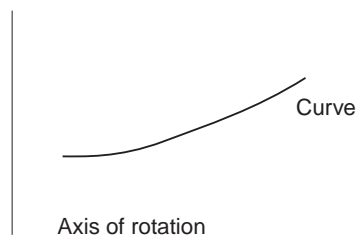


FIG. 5.8. *Another example.*

**Step 1:** Put down the  $a_1, a_2, a_3, \dots$  points. I am doing them in a different order than the last time:

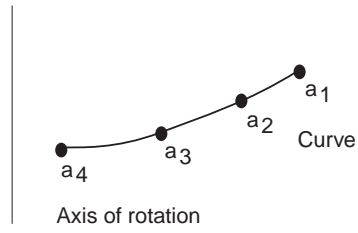


FIG. 5.9. *Put down points  $a_1, \dots$ .*

**Step 2:** Rotate the points around the axis

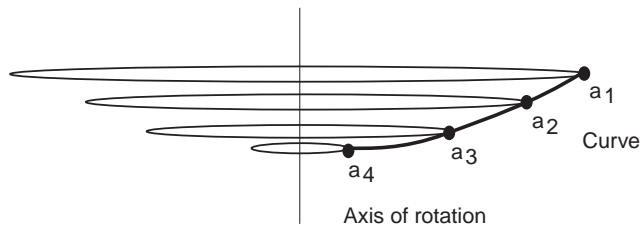


FIG. 5.10. *Rotate the points.*

**Step 3:** Mark the opposite points

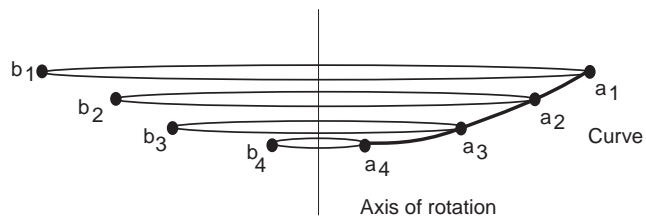


FIG. 5.11. *Mark the opposite points.*

**Step 4:** Join the opposite points in the same sequence as the original points:

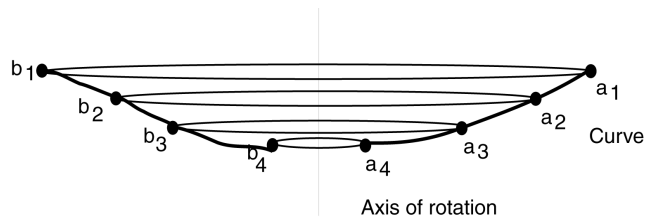


FIG. 5.12. *Join the marked points.*

**Step 5:** Slide the curve around the rings to visualize the surface

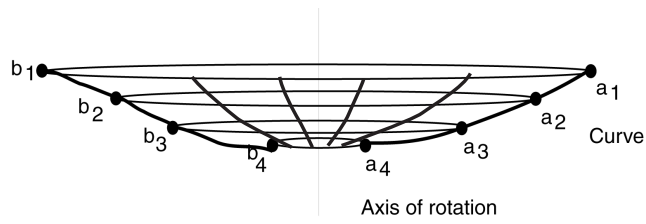


FIG. 5.13. *Slide the curve around.*

**Step 6:** If you erase all the construction details in your mind, you should see the surface as a bowl:

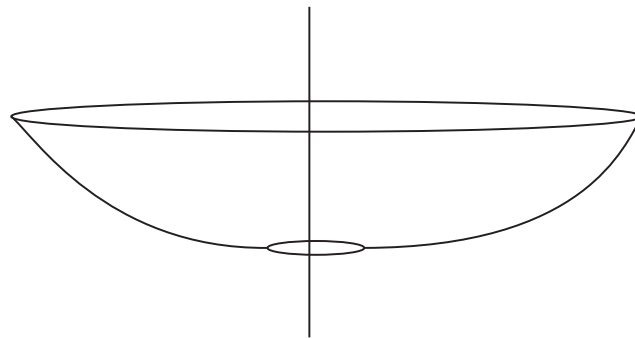


FIG. 5.14. *Surface of rotation.*



## 5.2. Common pitfalls.

1. When the curve intersects the axis, you should mark one of the points  $a_1, a_2, \dots$  at the intersection. And remember that the intersection point generates a ring of radius zero, i.e. it generates a point. There is an example below which shows some of the main steps in rotating such a curve:

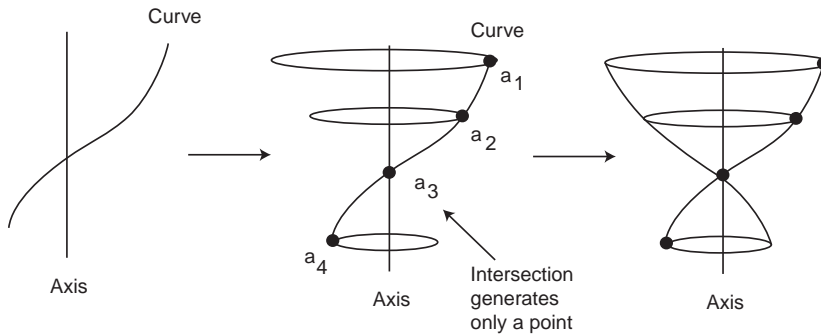


FIG. 5.15. *Curves intersect the axis.*

In this example, the surface of rotation is hour-glass shaped.

**6. Curves and axis are given by equations.** So far we just drew the curve and the axis by hand. We now turn to problems where the curve and axis are given by equations. For example, the curve is the graph of the function  $y = 1 + x^2$  for  $1 \leq x \leq 2$  while the axis is the line  $x = -1$ . The following figure shows this:

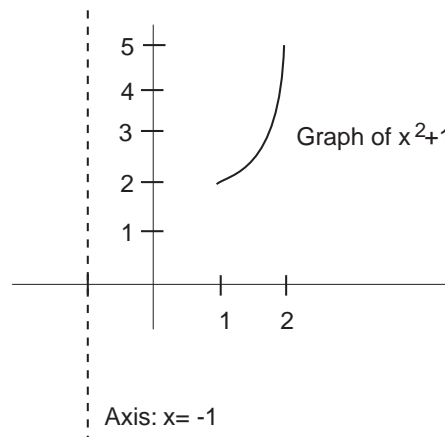


FIG. 6.1. *Curve and axis given by equations.*

Notice besides the curve and axis of rotation, I have drawn the coordinate axis on the figure (you have to draw the coordinate axis to graph the curve and the axis of rotation). The biggest problem with such diagrams is that students tend to rotate the curve around the coordinate axis. This is a mistake. To avoid this, *you should mentally erase everything from the figure except the curve and the axis of rotation.*

After that you proceed to rotate the curve as before:

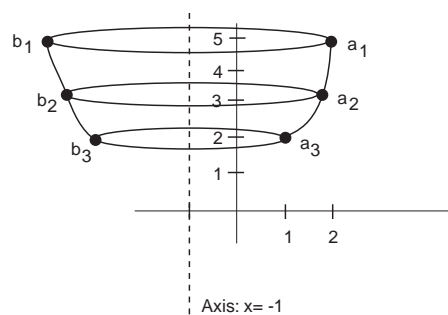


FIG. 6.2. *Rotating the curve.*

Just for emphasis I am drawing the figure with the curve rotated around the  $y$  axis (which is what many students tend to do). *This is wrong.*

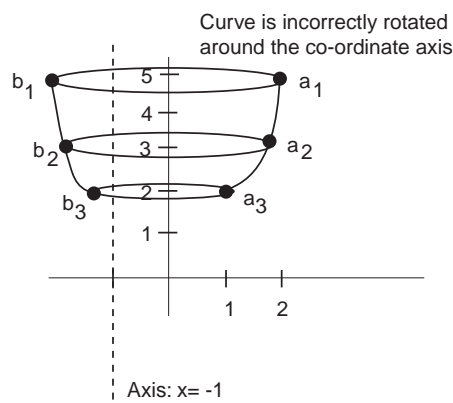


FIG. 6.3. *Rotating the curve.*

If you have trouble with coordinate axis try drawing the figure with a pencil and erasing the coordinate axis before you rotate the curve.

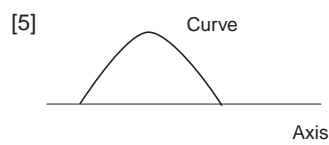
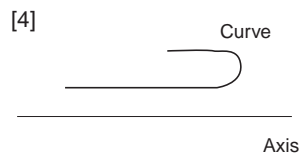
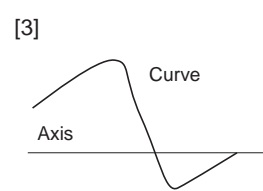
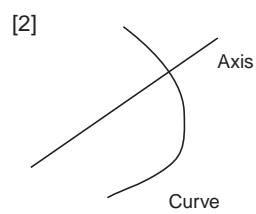
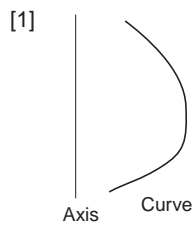
**7. The end.** That's it. I hope you have learned how to rotate a curve around an axis and visualize the resulting surface. With sufficient practice you should be able to visualize the rotated curve without the elaborate construction given here.

This is the first version of this tutorial and there are probably some things that are unclear in it. If you have any suggestions, please let me know. Also, let me know if you think something more should be added to this tutorial or I should make a section more elaborate.

Finally, if you are still having trouble visualizing rotated curves, please come to the tutorials.

## Exercises

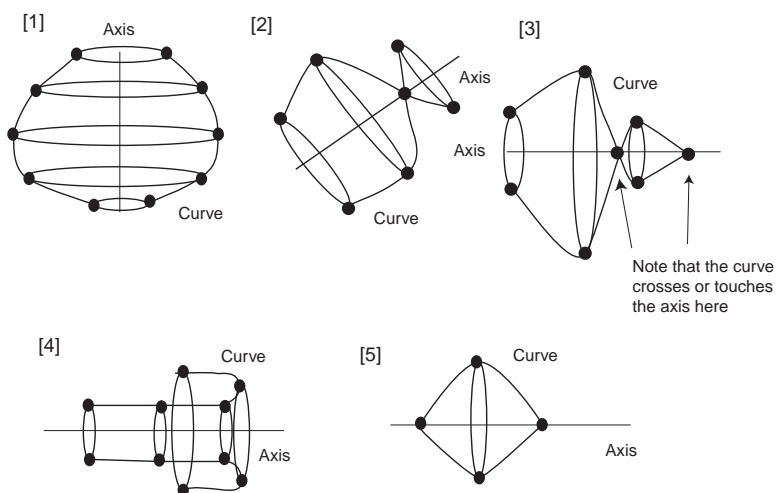
[A] Rotate each of the following curves around the given axis. Please try them yourself before you look at the answers (which are on the next page).



[B] Rotate the graph of  $y = x^3$  for  $-1 \leq x \leq 1$  around (1)  $x = 1$ , (2)  $y = 1$ .

# Answers

[A]



[B]

